# Bachelor of Science

## (B.Sc.)

## Differential Equations

## Semester-III

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# Contents

# Chapter 1

# Introduction to Differential Equations

Many of the general laws of nature – in physics, chemistry, biology, and astronomy – find their most natural expression in the language of differential equations. In this chapter we introduce differential equations. We also describe mathematical modeling, direction fields and solution of the differential equations.

## 1.1 Definitions

- An equation involving one dependent variable and its derivatives with respect to one or more independent variables is called **differential equation.**

- A differential equation involving only one independent variable, and hence only ordinary derivatives, is called **ordinary differential equation** (O. D. E.).
- A differential equation involving more than one independent variable, and hence partial derivatives, is called **partial differential equation (P. D. E.).**
- The **order** of a differential equation is the order of highest derivative occurring in it.
- The **degree** of a differential equation is the degree of the highest derivative occurs in it, *after* the differential equation has been cleared of radicals so far as derivatives are concerned.

***Example* 1** Determine the order and degree of the following differential equations. Also identify the partial differential equations.

1. $\frac{dy}{dt} = \cos t$
2. $(3x+2y)\frac{dy}{dx} + \left(7x^2 - y\right) = 0$
3. $y'' + 9y = 0$
4. $y\left(\frac{dy}{dx}\right)^2 + 2x = 0$
5. $\frac{d^2y}{dt^2} - \left[1 + \left(\frac{dy}{dt}\right)^2\right]^{\frac{3}{2}} = 0$
6. $y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0$
7. $t^3 y''' y' + 2e^t y'' = (t^2+2)y^2$
8. $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$
9. $y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} - z^2\frac{\partial u}{\partial y} = 0$
10. $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = k\,z.$

*Answer:*

| | *Order* | *Degree* | | *Order* | *Degree* |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 6 | 1 | 2 |
| 2 | 1 | 1 | 7 | 3 | 1 |
| 3 | 2 | 1 | 8 | 2 | 1 |
| 4 | 1 | 2 | 9 | 1 | 1 |
| 5 | 2 | 2 | 10 | 1 | 1 |

Also, differential equations given in 8, 9 and 10 are partial differential equations.

## 1.2 Differential Equations Associated with Primitives

We recall that **primitive** is a relation or equation between the variables which involves arbitrary constants.

***Example* 2** Obtain the differential equation associated with the primitive $y = At^2 + Bt + C$.

*Solution*

The given is a primitive with three arbitrary constants. We find a differential equation that has no arbitrary constants. Differentiating the given primitive successively, we get

$$\frac{dy}{dt} = 2At + B, \quad \frac{d^2y}{dt^2} = 2A, \quad \frac{d^3y}{dt^3} = 0.$$

Hence, $\frac{d^3y}{dt^3} = 0$ is the differential equation corresponding to the given primitive.

***Example* 3** Show that the differential equation of all parabolas having $x$-axis as the axis of symmetry is $y\left(\frac{d^2y}{dx^2}\right)+\left(\frac{dy}{dx}\right)^2=0$.

**Solution**

Let the equation of a parabola having the $x$-axis as the axis of symmetry be

$$y^2 = a(x+b).$$

Differentiating we get

$$2y\frac{dy}{dx} = a;$$

and differentiating once again we get

$$y\left(\frac{d^2y}{dx^2}\right)+\left(\frac{dy}{dx}\right)^2=0$$

and is the differential equation representing the parabolas having $x$-axis as the axis of symmetry.

## 1.3 Solution of First Order Ordinary Differential Equations

### Explicit and Implicit Solutions

**Definition** A function $y = g(t)$ is called a **solution** of a given first order differential equation on some interval, say, $a < t < b$ (perhaps infinite) if *g(t)* is defined and differentiable throughout that interval and is such that the equation becomes an identity when $y$ and $y\prime$ are replaced by $g$ and $g\prime$, respectively.

***Example*** **4** The function $y = g(t) = e^{2t}$ is a solution of the first order differential equation

$$\frac{dy}{dt} = 2y \text{ for all } t$$

because by differentiating we obtain

$$\frac{dg}{dt} = 2e^{2t}$$

and when $y$ and $y\prime$ are replaced by $g$ and $g\prime$ , respectively, we see that the equation

$$\frac{dy}{dt} = 2y$$

reduces to the identity

$$2e^{2t} = 2e^{2t}.$$

- Solution given in the form $y = g(t)$ as in the Definition above is called **explicit solution**. i.e., $y = g(t) = e^{2t}$ is an explicit solution of the differential equation $y\prime = 2y$.

- Some times a solution of a differential equation will appear as an implicit function, i.e., implicitly given in the form $G(t, y) = 0$; then it is called **implicit solution**.

***Example*** **5** $t^2 + y^2 - 1 = 0$ $(y > 0)$ is an implicit solution of the differential equation

$$y\frac{dy}{dt} = -t$$

on the interval $-1 < t < 1$, as differentiating $t^2 + y^2 - 1 = 0$ with respect to $t$ and a rearrangement gives $yy' = -t$.

**General Solution and Particular Solutions**

**Definitions** The solution of a first order ordinary differential equation, which contains one arbitrary constant (say $c$, which can take infinitely many values) is called the **general solution**. A solution obtainable from the general solution by giving particular value to the arbitrary constant $c$ is called a **particular solution**.

The geometrical representation of the general solution is an infinite family of curves called **integral curves.** Each integral curve is associated with a particular value of $c$ and is the graph of the solution corresponding to that value of $c$.

***Example* 6** $y = \sin t + c$, where $c$ as arbitrary, is a general solution of the differential equation

$$\frac{dy}{dt} = \cos t$$

Each of the functions

$$y = \sin t,\ y = \sin t + 3,\ y = \sin t - \frac{3}{5},\ y = \sin t - 4\sqrt{7}$$

is a (particular) solution of the given differential equation.

The geometrical representation of the general solution $y = \sin t + c$ is given in Fig. 1.1. It represents an infinite family of curves (called **integral curves)**. We note that only some of the curves are displayed in Fig. 1.1.

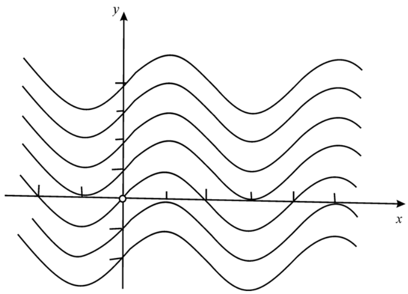


Figure 1.1:

***Example*** **7** $y = ce^t$ is a general solution and $y = -\frac{7}{2}e^t$ is a particular solution of the differential equation $\frac{dy}{dt} = y$.

The number of arbitrary constants depends on the order of the differential equation. If a differential equation is of order 2, its general solution would have two arbitrary constants. For example, we will see in a later chapter that the second order differential equation

$$\frac{d^2y}{dt^2} + 25y = 0$$

has the general solution

$$y = A\sin 5t + B\cos 5t$$

where $A$ and $B$ are arbitrary constants. We also note that a particular solution of the differential equation is

$$y = 3\sin 5t - \sqrt{2}\cos 5t.$$

### Singular Solutions

**Definition** (*Singular solution*) In some cases, there may be further solutions of a given differential equation, which cannot be obtained by assigning a definite value to the arbitrary constant in the general solution. Such a solution is called a **singular solution** of the differential equation.

***Example* 8** The differential equation

$$(y')^2 - xy' + y = 0$$

has the general solution $y = cx - c^2$, representing a family of straight lines, where each line corresponds to a definite value of $c$. A further a solution (that cannot be obtainable from the general solution and hence called singular solution) is $y = \frac{x^2}{4}$, representing a parabola. In this example, it can be seen that each particular solution represents a tangent to the parabola represented by the singular solution (Fig. 1.2).

***Example* 9** Solve the differential equation

$$\frac{dp}{dt} = 0.5p - 450. \tag{1.1}$$

Also give a particular solution.

*Solution*

To solve Eq. (1.1) we need to find functions $p(t)$ that, when substituted into the equation, gives an identity. First, rewrite Eq.

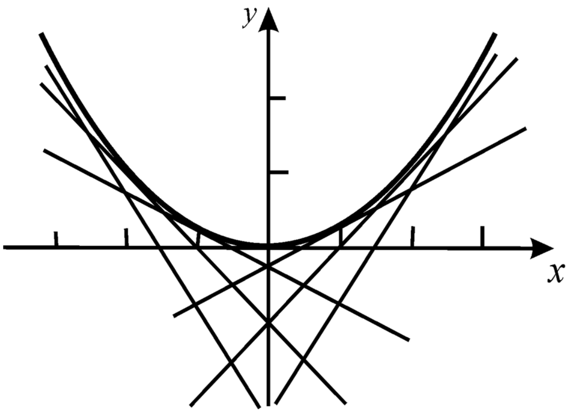


Figure 1.2: Singular solution and particular solutions of the differential equation $(y')^2 - xy' + y = 0$.

(1.1) in the form

$$\frac{dp}{dt} = \frac{p - 900}{2}. \tag{1.2}$$

Obviously $p = 900$ is a solution. To find other solutions, we note that if $p \neq 900$, Eq.(1.2) becomes

$$\frac{dp/dt}{p - 900} = \frac{1}{2}. \tag{1.3}$$

Letting $u = \ln|p - 100|$, and using the chain rule

$$\frac{du}{dt} = \frac{du}{dp} \cdot \frac{dp}{dt},$$

we have

$$\frac{d}{dt} \ln |p - 100| = \frac{1}{p - 100} \frac{dp}{dt}.$$

Hence Eq. (1.3) becomes

$$\frac{d}{dt}\ln|p-900| = \frac{1}{2}. \tag{1.4}$$

Then, integrating both sides of Eq.(1.4), we obtain

$$\ln|p-900| = \frac{t}{2} + C \tag{1.5}$$

where $C$ is an arbitrary constant of integration. Noting that $e^{\ln x} = x$ for $x > 0$, and by taking the exponential of both sides of Eq.(1.5), we find that

$$|p-900| = e^{(t/2)+C} = e^C e^{t/2}$$

or

$$p - 900 = \pm e^C e^{t/2}$$

or

$$p = 900 + c\,e^{t/2} \tag{1.6}$$

where $c = \pm e^C$ is also an arbitrary (nonzero) constant.

**Attention!** The solution $p = 900$ is a not a particular solution of (1.6), since $c$ is a nonzero arbitrary constant. The solution $p = 900$ can be included in the general solution if we take

$$p = 900 + c\,e^{t/2}$$

where $c$ is an arbitrary constant.

***Example* 10** Solve the differential equation

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}.$$

*Solution*

The given differential equation can be written as

$$\frac{dv}{dt} = -\frac{1}{5}(v - 49).$$

Obviously $v = 49$ is a solution. To find other solutions, we note that if $v \neq 49$,

$$\frac{dv/dt}{v - 49} = -\frac{1}{5}.$$

Letting $u = \ln|v - 49|$, and using the chain rule $\frac{du}{dt} = \frac{du}{dv}\frac{dv}{dt}$, we have

$$\frac{d}{dt}\ln|v - 49| = \frac{1}{v - 49} \cdot \frac{dv}{dt}.$$

Hence the given equation takes the form

$$\frac{d}{dt}\ln|v - 49| = -\frac{1}{5}.$$

Integrating both sides, we obtain

$$\ln|v - 49| = -\frac{t}{5} + C$$

Noting that $e^{\ln x} = x$ for $x > 0$, the above gives

$$|v - 49| = e^{-\frac{t}{5}+C}$$

or

$$v - 49 = \pm\, e^{C} e^{-t/5}$$

or

$$v = 49 + \; ce^{-t/5},$$

where $c = \pm e^{C}$ is an arbitrary (non zero) constant. If we allow $c$ to take 0 also; then $v = 49 + ce^{-t/5}$ includes all solutions.

## 1.4 Initial Value Problems

In many problems in Engineering or Physics one is not interested in the general solution of a given differential equation, but only in the particular solution *y(t)* satisfying a given initial condition, say, the condition that at some point $t_0$, the solution *y(t)* has a prescribed value $y_0$, viz., $y(t_0) = y_0$. Mathematically, an **initial value problem** (I.V.P.) of the first order differential equation is of the following form:

$y' = f(t, y)$, with the **initial condition** $y(t_0)= y_0$.

The **solution to the initial value problem is unique,** and it is the particular solution of the differential equation that satisfies the initial condition.

***Example* 11**

$$y' = \cos t;\ y(0) = 1$$

is an initial value problem. We have seen in Example 6 that

$$y = \sin t + c$$

is a general solution of $y' = \cos t$. Now, using the initial condition that $y(0)$ =1, i.e., $y = 1$ when $t = 0$, the general solution becomes

$1 = \sin 0 + c$, or $1 = 0 + c$ or $c = 0$.

Hence the particular solution that satisfies the initial condition is given by

$$y = \sin t + 1$$

and is the *unique solution* of the initial value problem.

**Attention!** Existence and uniqueness of solution of the initial value problem is guaranteed by the *Existence and Uniqueness Theorem* and the same will be discussed in a coming chapter (with $f(t,\ y) = \cos t$.)

## 1.5 Mathematical Models; Direction Fields

Most of the problems arising in Geometry, Physics or Engineering can be expressed in the form of differential equations. A differential equation that describes some physical process is often called a **mathematical model** of that process. **Mathematical Modeling** is the setting up of a mathematical model, i.e., formulating the problems of geometrical or physical type into mathematical terms by means of a differential equation, then finding general solution to the differential equation following which particular solution is found out.

***Example* 12** (*Geometrical problem*) A curve is defined by the condition that at each of its points $(t, y)$, its slope is equal to five times the abscissa of the point. Express this in terms of a differential equation.

*Solution*

It is given that the slope $\frac{dy}{dt}$ is equal to $5t$. Hence the required differential equation is $\frac{dy}{dt} = 5t$.

***Example* 13** (*Physical problem*) A particle of mass $m$ moves along the $y$-axis while subject to a force proportional to its displacement $y$ from a fixed point $O$ in its path and directed toward $O$. Express the condition by means of a differential equation.

*Solution*

Here force is given by $-ky$, where $k$ is the proportionality constant ($-$sign denotes the fact that the force acts in a direction opposite to the displacement).

Hence from the equation

mass x acceleration = Force,

we obtain the desired differential equation as

$$m\frac{d^2y}{dt^2} = -ky,$$

since the acceleration is given by $\frac{d^2y}{dt^2}$.

***Example* 14 (A Falling Object)** Suppose that an object is falling in the atmosphere near sea level. Formulate a differential equation that describes the motion.

*Solution*

1. We use $m$ to denote mass of the object, $t$ to denote time, and $v$ to represent the velocity of the falling object. We measure $m$ in kilograms, time $t$ in seconds and velocity $v$ in meters/second.

2. Since velocity changes with time, we think of $v$ as a function of $t$; in other words, $t$ is the independent variable and $v$ is the dependent variable. $v$ is taken to be **positive** in the downward direction-that is, when the object is falling.

3. By Newton's second law, the mass of the falling object times its acceleration is equal to the net force on the falling object. i.e.,

$$F = ma \tag{1.7}$$

   where $m$ is the mass of the falling object, $a$ is its acceleration, and $F$ is the net force exerted on the object. We measure $a$ in meters/second$^2$, and $F$ in Newtons. $a$ is related to $v$ by

$$a = \frac{dv}{dt},$$

so we can rewrite Eq.(1.7) in the form

$$F = m\frac{dv}{dt}. \tag{1.8}$$

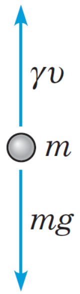


Figure 1.3:

The forces that act on the object as it falls are as follows:

1. Gravity exerts a force equal to the weight of the object, or $mg$, where $g$ is the acceleration due to gravity ($g$ is approximately equal to $9.8\,\mathrm{m/s^2}$ near the earth's surface.)

2. A force due to air resistance, or drag, that is proportional to the velocity, and has the magnitude $\gamma v$, where $\gamma$ is a constant called the drag coefficient (The physical units for $\gamma$ are mass/time, or kg/s for this problem).

Before writing an expression for the net force $F$, we note that gravity always acts in the downward (positive) direction, whereas drag acts in the upward (negative) direction. Thus

$$F = mg - \gamma v \tag{1.9}$$

and Eq.(1.9) then becomes

$$m\frac{dv}{dt} = mg - \gamma v \tag{1.10}$$

*Eq.(1.10) is a mathematical model of an object falling in the atmosphere near sea level.*

**Remark** The model in the previous example contains the three constants $m$, $g$, and $\gamma$; the constants $m$ and $\gamma$ depend on the particular object that is falling and they are usually different for different objects; $g$ is a physical constant, whose value is the same for all objects.

***Example* 15** Formulate a differential equation that describes the motion of an object falling in the atmosphere near sea level. Given $m = 5$ kg and $\gamma = 2.5$kg/s.

*Solution*

Proceeding as in the previous example, we obtain Eq.(1.10). Substituting $m = 5$ kg and $\gamma = 2.5$ kg/s in Eq. (1.10) and then on simplification, we obtain

$$\frac{dv}{dt} = 9.8 - \frac{v}{2}. \tag{1.11}$$

***Example* 16 (Field Mice and Owls)** Consider a population of field mice who inhabit a certain rural area.

(a) In the absence of predators we assume that the mouse population increases at a rate proportional to the current population. Set up a mathematical model denoting time by $t$ months and population by $p(t)$.

(b) Write the differential equation if the proportionality constant is 0.5 per month in (a) above.

(c) In addition to the problem in (b), suppose that several owls live in the same area and that they kill 15 field mice per day. Write the differential equation modeling this problem.

*Solution*

(a) In the absence of predators we assume that the mouse population increases at a rate proportional to the current population. If we denote time by $t$ and the mouse population by $p(t)$, then the assumption about population growth can be expressed by the equation

$$\frac{dp}{dt} = rp \tag{1.12}$$

where the proportionality factor $r$ is called the **rate constant** or **growth rate**.

(b) Suppose that time is measured in months and that the rate constant $r$ has the value 0.5/month. Then using (1.12),

$$\frac{dp}{dt} = 0.5p \tag{1.13}$$

Each term in Eq.(1.13) has the units of mice/month.

(c) Now we add to the problem in (b) by supposing that several owls live in the same neighborhood and that they kill 15 field mice per day (i.e., 450 field mice per month.) To bring this information into the model, we must add another term to the differential equation (1.13), so that it becomes

$$\frac{dp}{dt} = 0.5p - 450. \tag{1.14}$$

**Remark** A more general version of Eq. (1.14) is

$$\frac{dp}{dt} = rp - k \tag{1.15}$$

where the growth rate $r$ and the **predation rate** $k$ are unspecified.

**Geometrical Considerations, Isoclines**

Consider a differential equation in the *explicit form*

$$\frac{dy}{dt} = f(t, y)$$

where$f$ is a given function of two variables $t$ and $y$, and is called the **rate function.**

The explicit form has the geometrical interpretation that the slope of a solution $y = y(t)$ has the value $f(t_0, y_0)$ at the point $(t_0, y_0)$. This leads us to a useful **graphical method** for obtaining a rough picture of the particular solution of the differential equation. The procedure follows:

1. Evaluate the value of $f$ at each point of a rectangular grid.

2. At each point of the grid, a short line segment (called **lineal element**) is drawn whose slope is the value of $f$ at that

point. Thus each lineal element is tangent to the graph of the solution passing through that point.

3. In this way we obtain a field of lineal elements, called the **direction field (slope field)** of $y' = f(t, y)$. With the help of the lineal elements we can easily graph approximation curves to the (unknown) solution curves of $y' = f(t, y)$ and thus obtain a qualitatively correct picture of these solution curves.

### Procedure for Finding Direction Field

Consider a differential equation in the *explicit form*

$$\frac{dv}{dt} = f(t, v) \tag{1.16}$$

where the rate function $f$ is a given function of two variables $t$ and $v$. To find direction field we proceed as follows:

1. Suppose that the velocity $v$ has a certain given value$v_0$.

2. Then, by evaluating the right side of Eq. (1.16), we can find the corresponding value $f(t,\ v_0)$ of $\frac{dv}{dt}$ .

3. Display the above information graphically in the $tv$-plane by drawing short line segments with slope $f(t,\ v_0)$ at several points on the line $v = v_0$.

4. As in the above way we obtain a field of lineal elements, called the direction field (slope field).

We illustrate this method in the following example.

***Example* 17 (A Falling Object (continued))** Investigate the behavior of solution of the differential equation

$$\frac{dv}{dt} = 9.8 - \frac{v}{5} \tag{1.17}$$

without solving the differential equation.

*Solution*

Here

$$\frac{dv}{dt} = f(t,\ v)$$

where $f(t,\ v) = 9.8 - \frac{v}{5}$.

Suppose that the velocity $v$ has a certain given value. Then, by evaluating the right side of Eq. (1.17), we can find the corresponding value of $\frac{dv}{dt}$.

1. For instance, if $v = 40$, then $\frac{dv}{dt} = 9.8 - \frac{40}{5} = 1.8$. This means that the slope of a solution $v = v(t)$ has the value $f(t,\ 40) = 1.8$ at any point where$v = 40$. We display this information graphically in the $tv$-plane by drawing short line segments with slope 1.8 at several points on the line$v = 40$.

2. Similarly, if $v = 50$, then $\frac{dv}{dt} = 9.8 - \frac{50}{5} = -0.2$, so we draw line segments with slope $-0.2$ at several points on the line $v = 50$.

3. We obtain Fig. 1.4 by proceeding in the same way with other values of $v$. Fig. 1.4 is the direction field (slope field) of the

given differential equation.

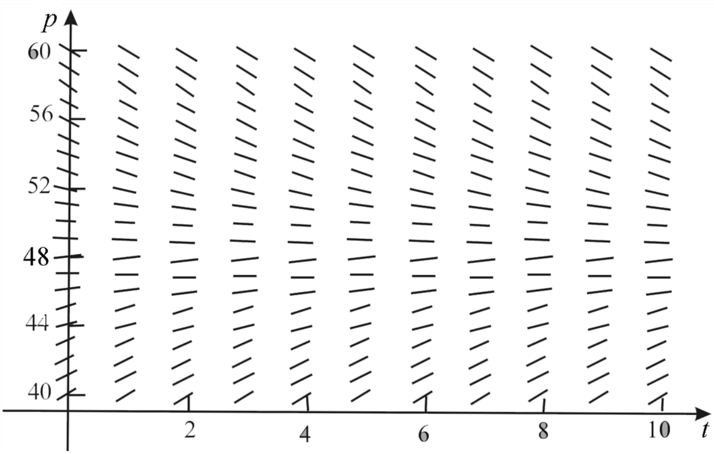


Figure 1.4: A direction field of Eq. (1.17): $\frac{dv}{dt} = 9.8 - \frac{v}{5}$

Now a solution of Eq. (1.17) is a function $v = v(t)$ whose graph is a curve in the $tv$-plane. The importance of the direction field in Fig. 1.4 is that each line segment is a tangent line to one of these solution curves. Thus, even though we have not found any solutions, and no graphs of solutions appear in the figure, we can draw some qualitative conclusions about the behavior of solutions:

1. For instance, if $v$ is less than a certain **critical value** (i.e., the value of $v$, where $\frac{dv}{dt} = 0$), then all the line segments have positive slopes, and the speed of the falling object increases as it falls.

2. If $v$ is greater than a certain critical value, then the line segments have negative slopes, and the speed of the falling object decreases as it falls.

Now $\frac{dv}{dt} = 0$ implies $9.8 - \frac{v}{5} = 0$ implies $v = (5)(9.8) = 49\,\text{m/s}$. It is the critical value of $v$ that separates objects whose speed is increasing from those whose speed is decreasing.

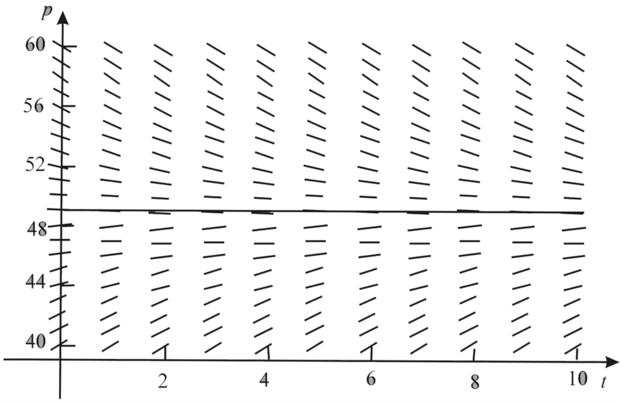


Figure 1.5: Direction field and equilibrium solution for Eq. 1.17

The constant function $v(t) = 49$ is a solution of Eq. (1.17), since substituting $v(t) = 49$(so that $\frac{dv}{dt} = 0$) into Eq. (1.17) makes each side of the equation (1.17) to zero. Being a constant function, the solution $v(t) = 49$ does not change with time, and is called an **equilibrium solution.** It is the solution that corresponds to a perfect balance between gravity and drag. The equilibrium solution $v(t) = 49$ is shown by superimposing on the direction field given in Fig. 1.5. From this figure we can conclude that all other solutions seem to be converging to the equilibrium solution as $t$ increases.

**Remark**: In an earlier example we have seen that the general solution of the differential equation

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}$$

is

$$v = v(t) = 49 + c\,e^{-\frac{1}{5}t}.$$

Also it can be seen that as $t \to \infty$, $e^{-\frac{1}{5}t} \to 0$ so that $v(t) \to 49$, the equilibrium solution.

***Example*** **18** Find the equilibrium solution of

$$m\frac{dv}{dt} = mg - \gamma v \ \ (m > 0,\ \gamma > 0).$$

*Solution*

$\frac{dv}{dt} = 0$ implies $mg - \gamma v = 0$ implies $v = \frac{mg}{\gamma}$.

Hence the equilibrium solution is

$$v = \frac{mg}{\gamma}.$$

***Example*** **19** Find the equilibrium solution of

$$\frac{dv}{dt} = rp - k. \tag{1.18}$$

*Solution*

$\frac{dv}{dt} = 0$ gives $rp - k = 0$ implies $p(t) = \frac{k}{r}$.

Hence the equilibrium solution of Eq.(1.18) is $p(t) = k/r$.

***Example* 20** Investigate the solutions of

$$\frac{dp}{dt} = 0.5p - 450 \tag{1.19}$$

graphically.

*Solution*

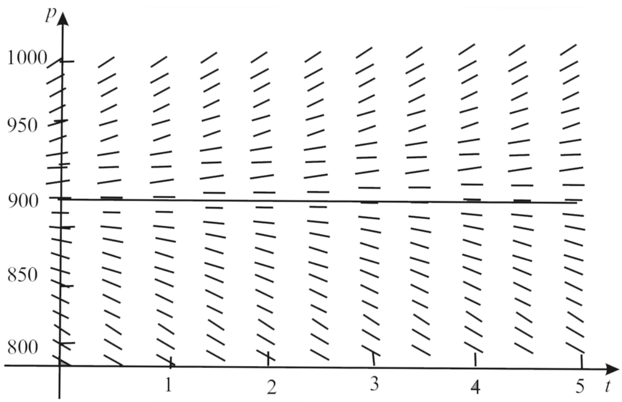


Figure 1.6: Direction field and equilibrium solution for Eq. (1.19)

A direction field for Eq.(1.19) is shown in Fig. 1.6. For sufficiently large values of $p$ it can be seen from the figure or directly from Eq.(1.19) itself, that $dp/dt$ is positive, so that solutions increase. On the other hand, if $p$ is small, then $dp/dt$ is negative and solutions decrease. Again, the critical value of $p$, which separates solutions that increase from those that decrease, is the value of $p$ for which $dp/dt$ is zero. Now $dp/dt = 0$ in Eq.(1.19) gives the equilibrium solution

$$p(t) = 900$$

for which the growth term 0.5 and the term 450 in Eq.(1.19) are exactly balanced. The equilibrium solution is superimposed in the direction field shown in Fig. 1.6.

Chapter 2

# Solutions of Some Differential Equations

## 2.1 Solutions of Differential Equations

***Example* 1[ A Falling Object (continued)]** Consider a falling object of mass $m = 10$ kg and drag coefficient $\gamma = 2$ kg/s. Suppose the object is dropped from a height of 300m. Find its velocity at any time $t$. How long will it take to fall to the ground, and how fast will it be moving at the time of impact?

*Solution*

If the velocity is $v$, with the aid of Eq.(1.10) in the previous chapter, the differential equation corresponding to the given problem is

$$10\frac{dv}{dt} = 10 \times 9.8 - 2v.$$

i.e.,

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}. \tag{2.1}$$

The object is dropped means the initial velocity is zero, so we have the initial condition

$$v(0) = 0. \tag{2.2}$$

Writing (2.1) in the form

$$\frac{dv/dt}{v - (9.8)(5)} = -\frac{1}{5}.$$

and integrating both sides, we obtain

$$\ln|v(t) - 49| = -\frac{t}{5} + C$$

which gives

$$v(t) - 49 = e^{-\frac{t}{5}+C}$$

so that (by taking $c = e^C$) it follows that

$$v(t) = 49 + ce^{-\frac{t}{5}}.$$

That is, the general solution of Eq. (2.1) is

$$v = 49 + ce^{-t/5} \tag{2.3}$$

where $c$ is arbitrary. To determine $c$, we substitute $t = 0$ and $v = 0$ from the initial condition (2.2) into Eq. (2.3), and obtain

$c = -49$. Then the solution of the initial value problem is

$$v = 49(1 - e^{-t/5}). \tag{2.4}$$

Eq. (2.4) gives the velocity of the falling object at any positive time (before it hits the ground).

**Remark** Graphs of the solution (2.3) for several values of $c$ are shown in Fig.2.1, with the solution (2.4) shown by the heavy curve. It is evident that all solutions tend to approach the equilibrium solution $v = 49$. This confirms the conclusions we reached earlier on the basis of the direction fields in Fig.1.4 and Fig.1.5.

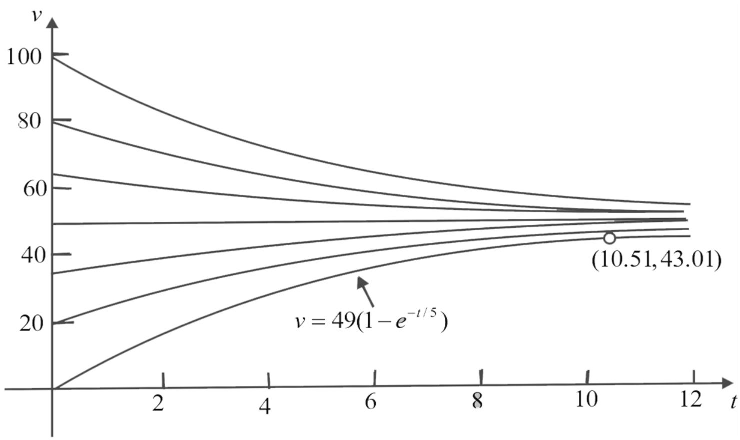


Figure 2.1:

To find the velocity of the object when it hits the ground, we need to know the time at which impact occurs. In other words, we need to determine how long it takes the object to fall 300 m. To do this, we note that the distance $x$ the object has fallen is related to its velocity $v$ by the equation $v = dx/dt$, or

$$\frac{dx}{dt} = 49(1 - e^{-t/5}). \tag{2.5}$$

Consequently, by integrating both sides of Eq.(2.5), we have

$$x = 49t + 245e^{-t/5} + c \tag{2.6}$$

where $c$ is an arbitrary constant of integration. The object starts to fall when $t = 0$, so we know that $x = 0$ when$t = 0$. From Eq.(2.6) it follows that $c = -245$, so the distance the object has fallen at time $t$ is given by

$$x = 49t + 245e^{-t/5} - 245. \tag{2.7}$$

Let $T$ be the time at which the object hits the ground; then $x = 300$ when $t = T$. By substituting these values in Eq.(2.7), we obtain the equation

$$300 = 49T + 245\ e^{-\frac{T}{5}} - 245$$

or

$$49T + 245e^{T/5} - 545 = 0. \tag{2.8}$$

The value of $T$ satisfying Eq. (2.8) can be approximated by a numerical process (for example, Newton-Raphson Method) using a scientific calculator or computer, with the result that $T \cong 10.51\,\mathrm{s}$.

At this time, the corresponding velocity $v_T$ is found from Eq. (2.4) to be $v_T \cong 43.01\,\text{m/s}$. The point (10.51, 43.01) is also shown in Fig. 2.1.

***Example* 2** (*Radioactivity, exponential decay*) Experiments show that a radioactive substance decomposes at a rate proportional to the amount present at time $t$. Suppose at time $t = 0$, 2 grams of a particular radioactive substance be present. Then what amount of the substance will be there at time $t$ $(t > 0)$.

*Solution*

**Step 1** (*Setting up a mathematical model of the physical process*)

Let $y(t)$ denote the amount of substance present at time $t$. It is given that rate of change $dy/dt$ is proportional to $y$. Thus

$$\frac{dy}{dt} = ky,$$

where $k$ is the proportionality constant of negative sign, which is a constant depends only on the nature of the radioactive substance. (For example, if the radioactive substance is radium, then $k \approx -1.4 \times 10^{-11}\,\text{sec}^{-1}$)

**Step 2** (*Solving the differential equation*)

The given differential equation, by separating variables (a detailed discussion will be made in the next chapter), gives

$$\frac{dy}{y} = k.$$

Integrating with respect to $t$,

$$\ln\ y = kt + C$$

Hence

$$y(t) = ce^{kt},\ \text{with } c = e^C$$

is the general solution to the differential equation $\frac{dy}{dt} = ky$.

**Step 3** (*Determination of a particular solution using the initial conditions*) The particular solution satisfying the given initial condition can be obtained by finding out the particular value of $c$. Now by putting $t = 0$, $y = 2$ in $y(t) = ce^{kt}$ we get $c = 2$ and so the *unique solution* to the initial value problem is $y(t) = 2e^{kt}$.

Hence the amount of substance at time $t$ is given by $y(t) = 2e^{kt}$

***Example* 3** Find the curve through the point (1, 1) in the $xy$-plane having at each of its points the slope is $-y/x$.

*Solution*

**Step 1** (*Setting up a mathematical model of the geometrical problem*)

Here $y$ is a function of $x$ and it is given that the slope $dy/dx$ is $-y/x$. i.e., the differential equation is

$$\frac{dy}{dx} = -\frac{y}{x}.$$

**Step 2** (*Solving the differential equation*)

Separating the variables, we obtain

$$\frac{dy}{y} = -\frac{dx}{x}.$$

Integration yields,

$$\ln y = -\ln x + \ln c$$

i.e.,

$$\ln y + \ln x = \ln c$$

i.e.,

$$yx = c$$

i.e.,

$$y = \frac{c}{x}$$

That is, the general solution to the differential equation $\frac{dy}{dx} = -\frac{y}{x}$ is $y(x) = \frac{c}{x}$, where $c$ is an arbitrary constant.

**Step 3** (*Determination of a particular solution*)

We want to find out the particular solution that passes through the point (1,1). Now by putting $x = 1$, $y = 1$ in $y(x) = \frac{c}{x}$ we get $c = 1$ and so the particular solution is $y(x) = \frac{1}{x}$.

***Example*** **4** Let $g$ denote the acceleration due to gravity (with value 9.8ms$^{-1}$) and $s(t)$ the distance of a freely falling body in vacuum at time $t$ sec. Set up a mathematical model (i.e., set up a differential equation) for the law that "the acceleration of a freely falling body in vacuum is the acceleration due to gravity." Also, show that the particular solution such that $s = 0$ at time $t = 0$ is

$$s(t) = \frac{1}{2}gt^2.$$

*Solution*

**Step 1** (*Setting up a mathematical model*)

By the law, acceleration $= g$,

i.e.,

$$\frac{d^2s}{dt^2} = g.$$

**Step 2** (*Solving the differential equation*)

Integrating the differential equation, we get

$$\frac{ds}{dt} = gt + A,$$

where $A$ is an arbitrary constant. Further integration yields,

$$s = \frac{1}{2}gt^2 + At + B,$$

where $B$ is also an arbitrary constant. This is the solution to the differential equation.

**Step 3** (*Determination of a particular solution*)

At time $t = 0$, $s = 0$. Also, since the body is freely falling, velocity is zero, so that $ds/dt = 0$. Using these values, equations in Step 2 gives

$A = 0$, $B = 0$,

and we obtain the particular solution

$$s(t) = \frac{1}{2}gt^2.$$

**Exercises**

In Exercises 1- 6, state the order of the given differential equation:

1. $y' + 5y = 0$
2. $y'' + 4y = 0$
3. $y' - y = 0$
4. $y''' = 6$
5. $y' + y \tan x = 0$
6. $y' - 3y^2 = 0$

In Exercises 7-11, verify that the given function is a solution of the differential equation given to the right of it.

7. $y = ce^{-8x}; \qquad y' + 8y = 0$
8. $y = c_1 e^x + c_2 e^{-x}; \qquad y'' - y = 0$
9. $y = x^4 + ax^2 + bx + c; \qquad y''' = 24x$
10. $y = A\cos x; \qquad y' + y\tan x = 0$
11. $y = A\cos 3x + B\sin 3x; \qquad y'' + 9y = 0$

In Exercises 12-15 , verify that the given function is a solution of the differential equation given to the right of it. Graph the corresponding curves for some values of the constant $c$.

12. $y = ce^{-x}; \qquad y' + y = 0$
13. $y = ce^{-x} + 4; \qquad y' + y = 4$

14. $y = cx^4$; $\quad xy' - 4y = 0$

15. $x^2 + y^2 = c$; $\quad yy' = -x$ In Exercises 16-19, verify that the given function is a solution of the differential equation given to the right of it and determine $c$ so that the resulting particular solution satisfies the given condition. Graph this particular solution.

16. $y = 3x + c$; $\quad y' = 3$; $\quad y = 1$ when $x = 0$.

17. $y = ce^{-x^2} + 2$; $\quad y' + 2xy = 0$; $\quad y = 0.5$ when $x = 0$.

18. $y = ce^{-x} + 2$; $\quad y' + y = 2$; $\quad y = 3.2$ when $x = 0$.

19. $y = x^3 + c$; $\quad y' = 3x^2$; $\quad y = -1$ when $x = 1$.

In Exercises 20-23, find a first order differential equation involving both $y$ and $y'$ for which the given function is a solution.

20. $y = \cos 2x 21. y = xe^{-x}$ 23. $y = x^3$

21. $y = -e^{-3x} 23. y = x^3 - 78$ 24. $y = \tan x$

22. $y = e^{-x^2}$ 25. $x^2 + y^2 = 16$.

Chapter 3

# First Order Linear Equations

In this chapter we discuss a method for the solution of first order ordinary linear differential equations.

## 3.1 Linear First Order Differential Equations

A first order differential equation of the form

$$\frac{dy}{dt} + p(t)y = g(t)$$

where $p(t)$ and $g(t)$ are functions of $t$ alone, is a **first order linear differential equation**.

If $g(t) = 0$ for every $t$, the equation is said to be **homogeneous**. Otherwise it is said to be **nonhomogeneous**.

**Determination of formula for the solution of the homogeneous linear differential equation**

Consider the general first order linear equation

$$\frac{dy}{dt} + p(t)y = g(t) \tag{3.1}$$

where $p$ and $g$ are given functions of $t$ alone.

By multiplying the differential equation (3.1) by a certain function $\mu(t)$, the resulting equation would be readily integrable. The function $\mu(t)$ is called an **integrating factor.**

To determine an appropriate integrating factor, we multiply Eq. (3.1) by an as yet undetermined function $\mu(t)$, obtaining

$$\mu(t)\frac{dy}{dt} + p(t)\mu(t)y = \mu(t)g(t). \tag{3.2}$$

By the product rule of differentiation, we have

$$\frac{d}{dt}\left[\mu(t)y\right] = \mu(t)\frac{dy}{dt} + \frac{d\mu(t)}{dt}y.$$

Hence the left side of Eq. (3.2) is the derivative of the product $\mu(t)y$, provided that $\mu(t)$ satisfies the equation

$$\frac{d\mu(t)}{dt} = p(t)\mu(t). \tag{3.3}$$

If we assume temporarily that $\mu(t)$ is positive, then we have

$$\frac{d\mu(t)/dt}{\mu(t)} = p(t),$$

and hence, integration yields

$$\ln \mu(t) = \int p(t)dt + k.$$

Now let $k = 0$. Then we obtain

$$\ln \mu(t) = \int p(t)dt.$$

Since $\exp(\ln x) = x$, for $x > 0$, and since our assumption is that $\mu(t) > 0$, the above gives

$$\mu(t) = \exp \int p(t)dt. \tag{3.4}$$

Returning to Eq.(3.2), we have

$$\frac{d}{dt}[\mu(t)y] = \mu(t)\, g(t). \tag{3.5}$$

Integrating, we obtain

$$\mu(t)\, y = \int \mu(t)g(t)\, dt + c \tag{3.6}$$

where $c$ is an arbitrary constant. Sometimes the integral in Eq. (3.6) can be evaluated in terms of elementary functions. However, in general this is not possible, so the general solution of Eq. (3.1)

is

$$y = \frac{1}{\mu(t)}\left[\int_{t_0}^{t} \mu(s)g(s)ds + c\right] \tag{3.7}$$

where $t_0$ is some convenient lower limit of integration. Observe that Eq.(3.7) involves two integrations, one to obtain $\mu(t)$ from Eq.(3.4) and the other to determine $y$ from Eq.(3.7)

**Working Method for the Solution of Linear Differential Equations**

To find the general solution of the linear differential equation

$$\frac{dy}{dx} + p(t)y = g(t) \tag{3.8}$$

1. Find the integrating factor by the formula

$$\mu(t) = \exp \textstyle\int p(t)dt. \tag{3.9}$$

2. Multiply Eq.(3.8) by $\mu(t)$ and obtain

$$\frac{d}{dt}[\mu(t)y] = \mu(t)g(t) \tag{3.10}$$

3. Integrating the above, we obtain

$$\mu(t)\, y = \textstyle\int \mu(t)\, g(t)\, dt \,+ c. \tag{3.11}$$

***Example* 1** Solve the linear differential equation

$$\frac{dy}{dt} - y = e^{2t}.$$

*Solution*

Here $p(t) = -1, \quad g(t) = e^{2t}$. Hence the integrating factor is given by

$$\mu(t) = e^{\int p(t)\,dt} = e^{\int -dt} = e^{-t}.$$

Using (3.11), implicit solution is given by

$$ye^{-t} = \int e^{2t}e^{-t}dt + c.$$

The corresponding explicit solution is

$$y = e^{t}\int e^{t}dt + ce^{t}$$

i.e., $y(t) = e^{2t} + ce^{t}$

***Example* 2** Solve the differential equation

$$\sin 2t\frac{dy}{dt} = y + \tan t.$$

*Solution*

Given equation can be written as

$$\frac{dy}{dt} - \frac{y}{\sin 2t} = \frac{\tan t}{\sin 2t} = \frac{1}{2}\sec^2 t,$$

which is in the linear form. The integrating factor is given by

$$\mu(t) = e^{\int p(t)dt} = e^{-\frac{1}{2}\int \frac{1}{\sin t\cos t}dt} = e^{-\frac{1}{2}\int \frac{\sec^2 t}{\tan t}dt}$$

$$= e^{-\frac{1}{2}\log\tan t} = \frac{1}{\sqrt{\tan t}}.$$

Hence, using (3.11) solution is given by

$$y\frac{1}{\sqrt{\tan t}} = \frac{1}{2}\int \frac{\sec^2 t}{\sqrt{\tan t}}dt + c$$

i.e.,

$$\frac{y}{\sqrt{\tan t}} = \sqrt{\tan t} + c.$$

***Example* 3** At time $t = 0$ a tank contains $Q_0$ lb of salt dissolved in 100gal of water (Fig. 3.1). Assume that water containing $\frac{1}{4}$ lb of salt/gal is entering the tank at a rate of $r$ gal/min and that the well-stirred mixture is draining from the tank at the same rate. Assuming that salt is neither created nor destroyed in the tank set up the initial value problem that describes this flow process. Find the amount of salt $Q(t)$ in the tank at any time, and also find the limiting amount $Q_L$ that is present after a very long time. If $r = 3$ and $Q_0 = 2Q_L$, find the time $T$ after which the salt level is within 2% of $Q_L$. Also find the flow rate that is required if the value of $T$ is not to exceed 45 min.

### Solution

1. The rate of change of salt in the tank, $\frac{dQ}{dt}$, is equal to the rate at which salt is flowing in minus the rate at which it is flowing out.

2. The rate at which salt enters the tank is the concentration $\frac{1}{4}$ lb/galtimes the flow rate $r$ gal/min, or $\frac{r}{4}$ lb/ min.

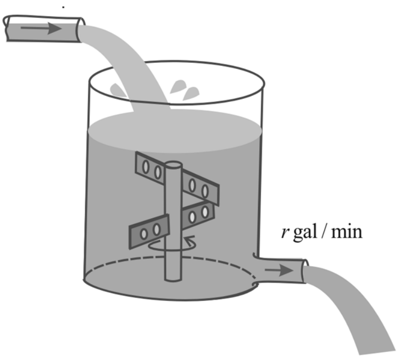


Figure 3.1:

3. To find the rate at which salt leaves the tank, we need to multiply the concentration of salt in the tank by the rate of outflow, $r$ gal/min.

4. Since the rates of flow in and out are equal, the volume of water in the tank remains constant at 100 gal, and since the mixture is 'well-stirred', the concentration throughout the tank is the same, and is $\frac{Q(t)}{100}$ lb/gal. Therefore the rate at which salt leaves the tank is $\frac{r\,Q(t)}{100}$ lb/min.

From the above information, the differential equation governing the given process is

$$\frac{dQ}{dt} = \frac{r}{4} - \frac{rQ}{100}. \tag{3.12}$$

The initial condition is

$$Q(0) = Q_0. \tag{3.13}$$

Rewriting Eq. (3.12) in the standard form for a linear equation, we have

$$\frac{dQ}{dt} + \frac{rQ}{100} = \frac{r}{4}$$

Thus the integrating factor is $e^{rt/100}$and the general solution is

$$Q(t) = 25 + ce^{-rt/100}$$

where $c$ is an arbitrary constant. To satisfy the initial condition (3.13), we must choose $c = Q_0 - 25$. Therefore the solution of the initial value problem (3.12), (3.13) is

$$Q(t) = 25 + (Q_0 - 25)e^{-rt/100} \tag{3.14}$$

or

$$Q(t) = 25(1 - e^{-rt/100}) + Q_0 e^{-rt/100} \tag{3.15}$$

From Eq.(3.14) or (3.15), we can see that $Q(t) \to 25$(lb)as $t \to \infty$, so the limiting value $Q_L$is 25.

Now suppose that $r = 3$ and $Q_0 = 2Q_L = 50$; then Eq.(**??**) becomes

$$Q(t) = 25 + 25e^{-0.03t} \tag{3.16}$$

Since 2% of 25 is 0.5, we wish to find the time $T$ at which $Q(t)$ has the value 25.5. Substituting $t = T$ and $Q = 25.5$ in Eq. (3.16) and solving for $T$, we obtain[1]

$$T = (\ln 50)/0.03 \cong 130.4 \text{ min}$$

To determine the value of $r$ at $T = 45$, we use Eq. (3.14), set $t = 45,\ Q_0 = 50,\ Q(t) = 25.5$,and solve for $r$. We obtain

$$r = (100/45) \ln 50 \cong 8.69 \,\text{gal}/\text{min}$$

***Example* 4** Consider a pond that initially contains 10 million gal of fresh water. Water containing an undesirable chemical flows into the pond at the rate of 5 million gal/yr, and the mixture in the pond flows out at the same rate. The concentration $\gamma(t)$ of chemical in the incoming water varies periodically with time according to the expression $\gamma(t) = 2 + \sin 2t \,\text{g/gal}$. Construct a mathematical model of this flow process and determine the amount of chemical in the pond at any time.

*Solution*

1. Since the incoming and outgoing flows of water are the same, the amount of water in the pond remains constant at $10^7$ gal.

2. Let us denote time by $t$, measured in years, and the chemical by $Q(t)$, measured in grams. Then the rate of change of chemical in the pond, $\frac{dQ}{dt}$, is equal to the rate at which the chemical flows into the pond minus the rate at which it is flowing out.

[1] Note that ln 50 is the natural logarithm of 50.

(a) The rate at which the chemical flows in is

$$\gamma \times (5 \times 10^6)\,\text{gal/yr} = (5 \times 10^6)\,\text{gal/ yr}\ (2 + \sin\ 2t)\,\text{g/gal}$$

(b) The concentration of chemical in the pond is $\frac{Q(t)}{10^7}$ g/gal, so the rate of flow out is

$$(5 \times 10^6)\,\text{gal/yr}\,[Q(t)/10^7]\,\text{g/gal} = \frac{Q(t)}{2}\text{g/yr}$$

From the above information, we obtain the differential equation

$$\frac{dQ}{dt} = (5 \times 10^6)(2 + \sin 2t) - \frac{Q(t)}{2},$$

where each term has the units of g/yr.

Let $q(t) = Q(t)/10^6$. Then

$$\frac{dq}{dt} = \frac{1}{10^6}\,\frac{dQ}{dt},$$

so

$$\frac{dQ}{dt} = 10^6\frac{dq}{dt}.$$

Substituting these values, we obtain

$$\frac{dq}{dt} + \frac{1}{2}q = 10 + 5\sin 2t \tag{3.17}$$

Initially, there is no chemical in the pond, so the initial condition is

$$q(0) = 0. \tag{3.18}$$

The integrating factor of the linear differential equation is $\mu(t) = e^{\int \frac{1}{2}dt} = e^{\frac{t}{2}}$. Multiplying Eq.(3.17) by this factor we obtain

$$\frac{d}{dt}\left(\mu(t)q(t)\right) = (10 + 5\,\sin\,2t)\,\mu(t).$$

Integrating, we obtain

$$\mu(t)\,q(t) = \int \mu(t)\,(10 + 5\,\sin\,2t)\,dt + C$$

i.e.,

$$e^{t/2}q(t) = \int (10 + 5\,\sin\,2t)\,e^{t/2}dt + C$$

i.e.,

$$e^{t/2}q(t) = 20e^{t/2} + 5\int\,\sin\,2te^{t/2}dt + C$$

Take

$$I = \int \sin\,2t\,e^{t/2}dt$$

and apply integration by parts,

$$\int uv' = uv - \int u'v$$

with $u = \sin\,2t$ and $v' = e^{t/2}$. Then

$$\begin{aligned} I &= \sin\,2t\frac{e^{t/2}}{\frac{1}{2}} - \int 2\cos\,\,2t\,\frac{e^{t/2}}{\frac{1}{2}}dt \\ &= 2\sin\,2te^{t/2} - 4\left\{\underbrace{\cos\,\,2t}_{u}\,\,\underbrace{e^{t/2}}_{v'}\,dt\right. \\ &\qquad \text{with new } u = \cos\,2t \text{ and } v' = e^{t/2} \end{aligned}$$

$$\begin{aligned} &= 2\sin\, 2te^{t/2} - 4\left[\cos\;\, 2t \cdot \frac{e^{t/2}}{\frac{1}{2}} - \int -\frac{\sin\, 2t}{2} \cdot \frac{e^{t/2}}{\frac{1}{2}}\, dt\right] \\ &= 2\sin\, 2t\, e^{t/2} - 8\,\cos\;\, 2t\, e^{t/2} - 16I. \end{aligned}$$

Hence

$$I = \frac{1}{17}\left[2\,\sin\, 2t\, e^{t/2} - 8\,\cos\, 2t\, e^{t/2}\right]$$

Substituting this and simplifying, we obtain the general solution

$$q(t) = 20 - \frac{40}{17}\cos 2t + \frac{10}{17}\sin 2t + ce^{-t/2}$$

Using the initial condition we obtain

$$0 = q(0) = 20 - \frac{40}{17} + c,$$

which gives

$$c = -300/17,$$

so the solution of the initial value problem (3.17) with (3.18) is

$$q(t) = 20 - \frac{40}{17}\cos 2t + \frac{10}{17}\sin 2t - \frac{300}{17}\, e^{-t/2}.$$

**Exercises**

In Exercises 1-10, find the general solution of the differential equation.

1. $\frac{dy}{dt} - y = 3$
2. $y' + 2ty = 0$
3. $y' + 2y = 6e^t$
4. $y' - 4y = 2t - 4t^2$

5. $y' + y = \sin t$

6. $y' + y = \cos t$

7. $y' + ky = \cos t$

8. $y' = (y-1)\cot t$

9. $ty' - 2y = t^3 e^t$

10. $t^2 y' + 2ty = \sinh 3t$

In Exercises 11-23, solve the given initial value problem.

11. $y' - y = e^t, y(1) = 0$

12. $y' + y = (t+1)^2, y(0) = 0$

13. $y' - 2y = 2\cosh 2t + 4\ ,\quad y(0) = -1.25$

14. $ty' - 3y = t^4(e^t + \cos t) - 2t^2\ ,\ y(\pi) = \pi^3 e^\pi + 2\pi^2$

15. $y' - y\cot t = 2t - t^2 \cot t, y(\frac{\pi}{2}) = \frac{\pi^2}{4} + 1$

16. $y' - y = 2te^{2t}, y(0) = 1$

17. $y' + 3y = te^{-3t}, \quad y(1) = 0$

18. $ty' + 2y = t^2 - t + 1, \quad y(1) = \frac{1}{2}, \quad t > 0$

19. $y' + (2/t)y = (\cos t)/t^2, \quad y(\pi) = 0, \quad t > 0$

20. $y' - 4y = e^{4t}, \quad y(0) = 2$

21. $ty' + 2y = \sin t, \quad y(\pi/2) = 1, \quad t > 0$

22. $t^3 y' + 4t^2 y = e^{-t}, \quad y(-1) = 0, \quad t < 0$

23. $ty' + (t+1)y = t, \quad y(\ln\ 2) = 1, \quad t > 0$

24. Consider a tank used in certain hydrodynamic experiments. After one experiment the tank contains 150 L of a dye solution with a concentration of 1 g/L. To prepare for the next experiment, the tank is to be rinsed with fresh water flowing in at a rate of 2 L/min. the well-stirred solution flowing out at the same rate. Find the time that will elapse before the concentration of dye in the tank reaches 1% of its original value.

25. A tank originally contains 100 gal of fresh water. Then water containing $\frac{1}{2}$ lb of salt per gallon is poured into the tank at a rate of 2 gal/min, and the mixture is allowed to leave at the same rate of 2 gal/min, with the mixture again leaving at the same rate. Find the amount of salt in the tank at the end of an additional 10 min.

26. Suppose that a sum $S_0$ is invested at an annual rate of return r compounded continuously.

27. Find the time $T$ required for the original sum to double in value as a function of r.

28. Determine $T$if $r = 8\%$

29. Find the return rate that must be achieved if the initial investment is to double in 8 years.

30. Raju borrows ‘ 8000to buy a motor bike. The lender charges interest at an annual rate of 10%. Assuming that interest is

compounded continuously and Raju makes payments continuously at a constant annual rate $k$, determine the payment rate $k$ that is required to pay off the loan in 3 years. Also determine how much interest is paid during the 3-year period.

**Answers**

1. $y = c\,e^{t} - 3$
2. $ye^{t^2} = c$
3. $y = c\,e^{-2t} + 2\,e^{t}$
4. $y = t^2 + ce^{4t}$
5. $y = c\,e^{t} + \frac{1}{2}(\sin t - \cos t)$
6. $y = \frac{1}{5}(2\cos t + \sin t) + c\,e^{-2t}$
7. $y = (c + t)\,e^{-4t}$
8. $y = 1 + c\sin t$
9. $y = c\,t^2 + t^2\,e^{t}$
10. $y = (\frac{1}{3}t^2)(\cosh 3t + c)$
11. $y = (t + 1)e^{t}$
12. $y = -e^{-t} + t^2 + 1$
13. $y = (t + 1)\,e^{2t} - \frac{1}{4}e^{-2t} - 2$
14. $y = [e^{t} + \sin t\,]\,t^3 + 2t^2$

15. $y = t^2 + \sin t$

16. $y = 3e^t + 2(t-1)e^{2t}$

17. $y = (t^2-1)\frac{e^{-3t}}{2}$

18. $y = \frac{3t^4-4t^3+6t^2+1}{12t^2}$

19. $y = \frac{\sin t}{t^2}$

20. $y = (t+2)e^{4t}$

21. $y = t^{-2}\left[\frac{\pi^2}{4} - 1 - t\cos t + \sin t\right]$

22. $y = -\frac{(1+t)e^{-t}}{t^4}, \ t \neq 0$

23. $y = \frac{t-1+2e^{-t}}{t}, \ t \neq 0$

24. $t = 75 \ \ln \ \ 100 \ \text{min} \ \cong 345.4 \ \text{min}$

25. $Q = 50e^{-0.2}(1-e^{-0.2})\,\text{lb} \ \cong 7.42\,\text{lb}$

26. (a) $\frac{\ln 2}{r}$ yr (b) 8.66 yr (c) 8.66

27. $k = 3086.64\,/\text{yr}\ 3 \times 3086.64 - 8000 \ =$ ‘ 1259.92

## 3.2 Method of Variation of Parameters

**Method of variation of parameter (M.V.P)** is an alternative method for finding the general solution of the linear differential equation

$$y' + p(t)y = g(t). \tag{3.19}$$

A solution of the corresponding homogeneous equation (i.e., an equation with $g(t) \equiv 0$) is

$$y(t) = Ae^{-\int p(t)dt} \tag{3.20}$$

Using this function, let us try to find out a function $A(t)$ such that

$$y(t) = A(t)e^{-\int p(t)dt} \tag{3.21}$$

is the general solution of the nonhomogeneous linear equation in (3.19). This attempt is suggested by the form of the general solution $Ay(t)$of the homogeneous equation and consists in replacing the parameter $A$ by a variable $A(t)$. Therefore, this approach is called the **method of variation of parameters**.

**Working Method for solving by method of variation of parameters**

To solve the linear differential equation

$$y' + p(t)y = g(t)$$

(i) First find the solution $y(x)$of the corresponding homogeneous equation

$$y' + p(t)y = 0$$

by the formula

$$y(t) = Ae^{-\int p(t)dt}$$

(ii) Then replace $A$ by $A(t)$ and find the function $A(t)$ by substituting

$$y(t) = A(t)e^{-\int p(t)dt}$$

in the non-homogeneous linear differential equation.

***Example* 5** Applying method of variation of parameters solve

$$\frac{dy}{dt} - y = 3e^t.$$

*Solution*

Here $p(t) = -1, \quad g(t) = 3e^t$.

Hence the general solution to the corresponding homogeneous equation is given by

$$y(t) = Ae^{-\int p(t)dt} = Ae^{-\int -1dt} = Ae^t.$$

Replacing $A$ by $A(t)$, and assuming that $y(t) = A(t)e^t$ is the general solution of the nonhomogeneous equation, our next aim is find the function $A(t)$, by substituting $y(t) = A(t)e^t$ in the given nonhomogeneous differential equation, which gives

$$\frac{dA(t)}{dt}e^t + A(t)e^t - A(t)e^t = 3e^t$$

or

$$\frac{dA(t)}{dt}e^t = 3e^t$$

or

$$\frac{dA(t)}{dt} = 3.$$

Integrating, we obtain

$$A(t) = 3t + c.$$

Hence the general solution to the given differential equation is

$$y = (3t + c)e^t.$$

**Exercises**

Solve the following equations by the method of variation of parameters:

1. $y' - y = t$
2. $ty' + 2y = \frac{\sin t}{t}$
3. $y' = e^{2t} + y - 1$
4. $y' + 2y = e^{-2t}$
5. $ty' - 2y = t^4$
6. $(t + 4)y' + 3y = 3$
7. $y' - 2y = t^2 e^{2t}$
8. $y' + \frac{y}{t} = 3 \cos 2t, \quad t > 0$
9. $ty' + 2y = \cos t, \quad t > 0$
10. $2y' + y = 3t^2$
11. $y' - y = e^{2x}$

***Answers***

1. $y = c\, e^t - t - 1$
2. $t^2 y + \cos t = c$
3. $y = e^{2t} + 1 + c\, e^t$
4. $y = (t + c)e^{-2t}$
5. $\frac{1}{2}\, t^4 + c\, t^2$
6. $y = 1 + \frac{c}{(t+4)^3}$
7. $y = ce^{2t} + \frac{t^3 e^{2t}}{3}$
8. $y = \frac{c}{t} + \frac{3 \cos 2t}{4t} + \frac{3 \sin 2t}{2}$
9. $y = \frac{c + \cos t + t \sin t}{t^2}$
10. $y = ce^{-t/2} + 3t^2 - 12t + 24$
11. $y = e^{2x} + ce^x$

## 3.3 Bernoulli's Equation

A differential equation of the form

$$y' + p(t)y = g(t)y^n \tag{3.22}$$

where $p(t)$ and $g(t)$ are functions of $t$ alone is called **Bernoulli's equation**. (Note that here $n$ may be any real number). We can reduce equation (3.22) to the linear form as follows:

Dividing both sides of (3.22) by $y^n$, we get

$$y^{-n}y' + p(t)y^{1-n} = g(t)$$

Putting $y^{1-n} = z$, this equation becomes

$$\frac{1}{1-n}z' + p(t)z = g(t)$$

i.e.,

$$z' + (1-n)p(t)z = g(t)(1-n) \tag{3.23}$$

Now (3.23) is in the linear form, solution of which is familiar to us. After solving for $z$ in (3.23), using $y^{1-n} = z$, we get the value of $y$, i.e., the solution of (3.22).

***Example* 6** Solve $t\frac{dy}{dt} + y = ty^3$.

*Solution*

Dividing the given differential equation by $y^3$, we obtain

$$\frac{t}{y^3}\frac{dy}{dt} + \frac{1}{y^2} = t.$$

Putting $\frac{1}{y^2} = z$, the equation becomes

$$-\frac{1}{2}t\frac{dz}{dt} + z = t.$$

i.e.,

$$\frac{dz}{dt} - \frac{2}{t}z = -2.$$

The above is a linear differential equation and its integrating factor is

$$\mu(t) = e^{\int p(t)dt} = e^{\int -\frac{2}{t}dt} = e^{-2\ln|t|} = e^{\ln|t|^{-2}} = |t|^{-2} = \frac{1}{t^2}.$$

Hence the solution is given by

$$z\frac{1}{t^2} = \int -2\frac{1}{t^2}dt + c$$

or

$$\frac{z}{t^2} = \frac{2}{t} + c.$$

Now substituting $z = \frac{1}{t^2}$, the solution of the given differential equation is

$$(2 + ct)ty^2 = 1.$$

We can solve certain non linear first order ordinary differential equation in a similar fashion as we solve a Bernoulli equation. This is illustrated in the following Example.

***Example* 7** Solve $\frac{dy}{dt} + t\sin 2y = t^3\cos^2 y$.

*Solution*

Dividing by $\cos^2 y$, we obtain

$$\sec^2 y\frac{dy}{dt} + 2t\tan y = t^3$$

Put $\tan y = z$. Then

$$\sec^2 y\frac{dy}{dt} = \frac{dz}{dt}$$

and the equation becomes the linear differential equation

$$\frac{dz}{dt} + 2tz = t^3.$$

Now the integrating factor is given by

$$\mu(t) = e^{\int p(t)dt} = e^{\int 2t\,dt} = e^{t^2}.$$

So the solution is

$$ze^{t^2} = \int t^3e^{t^2}dt + c$$

or

$$ze^{t^2} = \frac{1}{2}\int 2t\cdot t^2e^{t^2}dt + c$$

or

$$ze^{t^2} = \frac{1}{2}\int ue^u du + c,$$

where $u = t^2$. Hence

$$ze^{t^2} = \frac{1}{2}e^u(u-1) + c$$

or

$$ze^{t^2} = \frac{1}{2}e^{t^2}(t^2-1) + c$$

or

$$\tan y e^{t^2} = \frac{1}{2}e^{t^2}(t^2-1) + c$$

or

$$\tan y = \frac{1}{2}(t^2-1) + ce^{-t^2}.$$

**Exercises**

Solve the following equations:

1. $y' + ty = ty^{-1}$
2. $y' + t^{-1}y = t^{-1}y^{-2}$
3. $2ty' = 10t^3y^5 + y$
4. $y' + y = ty^{-1}$
5. $y' - y\tan t = \frac{\sin t \cos^2 t}{y^2}$
6. $y' = y\tan t - y^2 \sec t$
7. $ty' + y = y^2 \log t$
8. $y' + yt^{-1} = t^3y^4$
9. $y' + \frac{ty}{1-x^2} = ty^{1/2}$
10. $ty' + y^2t^{-1} = y$
11. $(1-t^2)y' - ty = t^2y^2$
12. $ty' + y = t^2y^2 \log t$
13. $t\frac{dy}{dt} + y = t^3y^6$
14. Solve the initial value problem $y' - yt^{-1} = \frac{1}{2}y^{-1}; \quad y(1) = 0.$

**Answers**

1. $y^2 = 1 + ce^{t^{-2}}$
5. $y^3 \cos^3 t = \frac{1}{2}\cos^6 t + c$
6. $(t + c)y = \sin t$
7. $y(\log t + 1) + cty = 1$
8. $(c - 3t)t^3y^3 = 1$
9. $\sqrt{y} + \frac{1}{3}(1 - t^2) = c(1 - t^2)^{1/4}$
10. $x = y(\log t + c)$
11. $(1 + ty) = y(1 - t^2)^{1/2}(c + \sin^{-1} t)$
12. $t^3y(1 - \log t) + cty = 1$
13. $(ty)^{-5} = \frac{5}{2}t^{-2} + c$
14. $t^2 - y^2 - t = 0.$

Chapter 4

# Separable Equations

Certain first order differential equations can be reduced to the form

$$N(y)\,dy = M(x)dx \tag{4.1}$$

by algebraic manipulations. An equation that can be brought to the form as in (4.1) is called an *equation with separable variables*, or a **separable equation**, because in (4.1) the variables $x$ and $y$ are *separated* so that terms involving $x$ appear only on the right and that of $y$ appears only on the left. By integrating both sides of (4.1), we obtain

$$\int N(y)\,dy = \int M(x)\,dx \ + \ c \tag{4.2}$$

where $c$ is an arbitrary constant.

If we assume that $f$ and $g$ are continuous functions, the integrals in (4.2) will exist, and by evaluating these integrals we obtain the general solution of the given differential equation.

## 4.1 Identifying and Solving Separable Equations

Consider the first order differential equation

$$\frac{dy}{dx} = f(x, y). \tag{4.3}$$

To identify when an equation of the above form is a separable equation, we first rewrite Eq.(4.3) in the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0. \tag{4.4}$$

It is always possible to do this by setting $M(x, y) = -f(x, y)$and $N(x, y) = 1$, but there may be other ways as well. **If it happens that $M$ is a function of $x$ only and $N$ is a function of $y$ only,** then Eq.(4.4) becomes

$$M(x) + N(y)\frac{dy}{dx} = 0. \tag{4.5}$$

such an equation is said to be **separable,** because if it is written in the differential form

$$M(x)dx + N(y)dy = 0, \tag{4.6}$$

then, terms involving each variable may be placed on opposite sides of the equation. The differential form (4.6) is also more symmetric and tends to suppress the distinction between independent and dependent variables.

We now **show that** a separable equation can be solved by integrating the functions $M$ and $N$. To show this, let $H_1$ and $H_2$ be any antiderivatives of $M$ and $N$, respectively. Thus

$$H_1'(x) = M(x), \qquad H_2'(y) = N(y), \tag{4.7}$$

and then Eq.(4.5) takes the form

$$H_1'(x) + H_2'(y)\frac{dy}{dx} = 0. \tag{4.8}$$

Since $H_2$ is a function of $y$ and $y$ is a function of $x$, by the Chain Rule for Functions of One Variable,[1] we have

$$\frac{d}{dx}H_2(y) = \frac{d}{dy}H_2(y)\frac{dy}{dx}.$$

[1] The Chain Rule for Functions of One Variable
Let $w = f(y)$ be a differentiable function of $y$ and $y = \varphi(x)$ be a differentiable function of $x$, then $w$ is a differentiable composite function of $x$ and the derivative $\frac{dw}{dx}$ could be calculated using the Chain Rule given by

$$\frac{dw}{dx} = \frac{dw}{dy} \cdot \frac{dy}{dx}.$$

As an example, using the Chain Rule, we find $\frac{dw}{dx}$, when $w = \cosh^{-1} y$, and $y = x^2$ as follows:

$$\frac{dw}{dx} = \frac{d}{dy}(\cosh^{-1} y)\frac{d}{dx}(x^2) = \frac{1}{\sqrt{y^2-1}} \cdot 2x = \frac{2x}{\sqrt{x^4-1}}.$$

Also noting that

$$H_2'(y) = \frac{d}{dy}H_2(y),$$

the above gives

$$H_2'(y)\frac{dy}{dx} = \frac{d}{dy}H_2(y)\frac{dy}{dx} = \frac{d}{dx}H_2(y). \tag{4.9}$$

Hence, we can write Eq.(4.8) as

$$\frac{d}{dx}H_1(x) + \frac{d}{dx}H_2(y) = 0.$$

i.e.,

$$\frac{d}{dx}[H_1(x) + H_2(y)] = 0 \tag{4.10}$$

Integrating Eq.(4.10), we obtain

$$H_1(x) + H_2(y) = c, \tag{4.11}$$

where $c$ is an arbitrary constant. Any differentiable function $y = \phi(x)$ that satisfies Eq.(4.11) is a solution of Eq.(4.5); in other words, Eq.(4.11) defines the solution implicitly rather than explicitly.

***Example* 1** Show that the differential equation

$$10y \cdot \frac{dy}{dx} + 3x = 0 \tag{4.12}$$

is separable. Illustrate the procedure discussed above by solving this differential equation.

*Solution*

The given differential equation can be written as

$$3x + 10y\frac{dy}{dx} = 0$$

Comparing with the standard equation,

$M(x) = 3x$ and $N(y) = 10y$.

Hence given is a separable equation.

Since $y$ is a function of $x$, by the chain rule

$$\frac{d}{dx}f(y) = \frac{d}{dy}f(y)\ \frac{dy}{dx} = f'(y)\frac{dy}{dx}.$$

Here $f'(y) = 10y$. Hence

$$f(y) = \frac{10y^2}{2} = 5y^2.$$

Also $3x$ is the derivative of $\frac{3x^2}{2}$. Hence (4.12) takes the form

$$\frac{d}{dx}\left(\frac{3x^2}{2}\right) + \frac{d}{dx}\ (5y^2) = 0$$

i.e.,

$$\frac{d}{dx}\left(\frac{3x^2}{2} + 5y^2\right) = 0$$

By integrating, we obtain

$$\frac{3x^2}{2} + 5y^2 = C.$$

### Method of Solving Separable Equation

In practice, Eq.(4.11) is usually obtained from Eq.(4.6) by integrating the first term with respect to $x$ and the second term with respect to $y$. This is illustrated in the following example.

***Example* 2** Solve the differential equation $\frac{dy}{dx} + 2xy = 0$.

*Solution*

The given equation can be written in the form

$$2xdx + \frac{dy}{y} = 0.$$

The above is in the variable separable form with

$M(x) = 2x$ and $N(y) = \frac{1}{y}$.

By integrating the first term of the differential equation with respect to $x$ and the second term with respect to $y$, we obtain

$$x^2 + \ln\, y = c$$

or $\ln\, y = -x^2 + c$.

Since $\exp \ln w = w$ (for $w > 0$), the above yields

or

$$y = e^{-x^2+c}.$$

The above can also be written as $y = Ce^{-x^2}$, where $C = e^c$ in an arbitrary constant.

***Example* 3** Solve the differential equation $\frac{dy}{dx} = 1 + y^2$.

*Solution*

The given differential equation can be written as

$$\frac{dy}{1+y^2} = dx,$$

or

$$-dx + \frac{dy}{1+y^2} = 0$$

which is in the variable separable form with

$$M(x) = -1 \text{ and } N(y) = \frac{1}{1+y^2}.$$

By integrating the first term of the differential equation with respect to $x$ and the second term with respect to $y$, we obtain

$$-x + \tan^{-1} y = c$$

or

$$\tan^{-1} y = x + c.$$

### Separable Equation with Initial Condition

The differential equation (4.5), together with an initial condition

$$y(x_0) = y_0 \tag{4.13}$$

form an initial value problem. To solve this initial value problem, we must determine the appropriate value for the constant $c$ in Eq.(4.11). We do this by setting $x = x_0$ and $y = y_0$in Eq.(4.11) with the result that

$$c = H_1(x_0) + H_2(y_0) \tag{4.14}$$

Substituting this value of $c$ in Eq.(4.11) and noting that

$$H_1(x) - H_1(x_0) = \int_{x_0}^{x} M(s)ds$$

and

$$H_2(y) - H_2(y_0) = \int_{y_0}^{y} N(s)ds,$$

we obtain

$$\int_{x_0}^{x} M(s)ds + \int_{y_0}^{y} N(s)ds = 0 \tag{4.15}$$

Equation (4.15) is an implicit representation of the solution of the differential equation (4.5) that also satisfies the initial condition (4.13). We note that, to determine an explicit formula for the solution, Eq. (4.15) must be solved for $y$ as a function of $x$.

***Example*** **4** Solve the initial value problem

$$ay' = b - ky; \quad y(0) = 0.$$

*Solution*

The given equation can be written in the form

$$-dx + \frac{a}{b - ky}\,dy = 0.$$

The above is in the variable separable form with

$M(x) = -1$ and $N(y) = \frac{a}{b-ky}$.

By integrating the first term of the differential equation with respect to $x$ and the second term with respect to $y$, we obtain

$$-x - \frac{a}{k} \ln |b - ky| = c,$$

or

$$-\frac{a}{k} \ln |b - ky| = x + c,$$

where $c$ is an arbitrary constant.

To find the particular solution that satisfies the given initial condition, we substitute $x = 0$, $y = 0$, and get the value of $c$ as

$$c = -\frac{a}{k} \ln |b| \,,$$

so that the unique solution to the initial value problem is given by

$$-\frac{a}{k} \ln |b - ky| = x - \frac{a}{k} \ln |b|$$

or

$$-\frac{a}{k} \ln \left| 1 - \frac{ky}{b} \right| = x$$

or

$$1 - \frac{ky}{b} = e^{-\frac{kx}{a}}$$

or

$$y = \frac{b}{k}(1 - e^{-\frac{kx}{a}}).$$

**Remark** If $f(x,\ y_0) = 0$ for some value $y_0$ and for all $x$, then the constant function $y = y_0$ is a solution of the differential equation

$$\frac{dy}{dt} = f(x,\ y).$$

For example,

$$\frac{dy}{dx} = \frac{(y-2)\ \sin 3x}{1+3y^2}$$

has the constant solution $y = 2$. Other solutions can be found by separating the variables and integrating.

***Example* 5** Solve the initial value problem

$$\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)}, \tag{4.16}$$

with the initial condition

$$y(0) = -1 \tag{4.17}$$

and determine the interval in which the solution exists.

*Solution*

The differential equation can be written as

$$2(y-1)dy = (3x^2+4x+2)dx.$$

Integrating the left side with respect to y and the right side with respect to $x$ gives

$$y^2 - 2y = x^3 + 2x^2 + 2x + c, \tag{4.18}$$

where $c$ is an arbitrary constant. To determine the solution satisfying the prescribed initial condition, we substitute $x = 0$ and $y = -1$in Eq.(4.18), obtaining $c = 3$. Hence the solution of the initial value problem is given implicitly by

$$y^2 - 2y = x^3 + 2x^2 + 2x + 3 \tag{4.19}$$

Since Eq.(4.19) is quadratic in $y$, and we obtain

$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4}. \tag{4.20}$$

Equation (4.20) gives two solutions of the differential equation, **only one of which, however, satisfies the given initial condition.**

Since the initial condition is $y(0) = -1$, i.e., $y = -1$ when $x = 0$, we have to take the solution corresponding to the minus sign in Eq.(4.20), so

$$y = \phi(x) = 1 - \sqrt{x^3 + 2x^2 + 2x + 4} \tag{4.21}$$

is the solution of the initial value problem given by (4.16) and (4.17). Finally to determine the interval in which the solution (4.21) is valid, we must find the interval in which the quantity under the radical is positive. The only real zero of this expression is $x = -2$, so the desired interval is $x > -2$.

**Attention!** Note that if the plus sign is chosen by mistake in

Eq.(4.20), then we obtain the solution

$$y = \psi(x) = 1 + \sqrt{x^3 + 2x^2 + 2x + 4} \qquad (4.22)$$

of the same differential equation that satisfies the initial condition $y(0) = 3$.

***Example* 6** A body of constant mass $m$ is projected away from the earth in a direction perpendicular to the earth's surface with an initial velocity $v_0$. Assuming that there is no air resistance, but taking into account the variation of earth's gravitational field with distance, find

(a) an expression for the velocity during the ensuing motion.

(b) the initial velocity that is required to lift the body to a given maximum attitude $\xi$ above the surface of the earth, and

(c) the least initial velocity for which the body will not return to the earth (this velocity is called the **escape velocity.)**

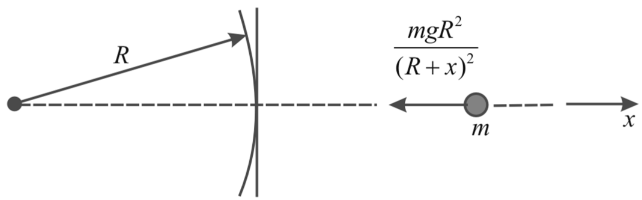


Figure 4.1:

*Solution*

Let the positive $y$-axis point away from the center of the earth along the line of motion with $y = 0$ lying on the earth's surface

(Fig. 4.1). The gravitational force acting on the body (that is, its weight) is inversely proportional to the square of the distance from the center of the earth and is given by

$$w(y) = -\frac{k}{(y+R)^2} \tag{4.23}$$

where $k$ is a constant, $R$ is the radius of the earth, and the minus sign signifies that $w(y)$ is directed in the negative $y$ direction. We know that on the earth's surface $w(0)$ is given by $-mg$, where $g$ is the acceleration due to gravity at sea level. Therefore (4.23) gives

$$-mg = w(0) = \frac{-k}{(0+R)^2}$$

or

$$k = mgR^2$$

and hence again by (4.23),

$$w(y) = -\frac{mgR^2}{(R+y)^2}.$$

Since there are no other forces acting on the body, the equation of motion is

$$m\frac{dv}{dt} = -\frac{mgR^2}{(R+y)^2} \tag{4.24}$$

and the initial condition is

$$v(0) = v_0 \tag{4.25}$$

By the chain rule for functions of one variable,

$$\frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = v\frac{dv}{dy}. \tag{4.26}$$

Hence Eq. (4.24) becomes

$$v\frac{dv}{dy} = -\frac{gR^2}{(R+y)^2}. \tag{4.27}$$

Separating the variables,

$$vdv = gR^2 \cdot \frac{dy}{(R+y)^2}.$$

Integrating,

$$\frac{v^2}{2} = \frac{gR^2}{R+y} + c. \tag{4.28}$$

Since $y = 0$ when $t = 0$, the initial condition (4.26) at $t = 0$ can be replaced by the condition that $v = v_0$ when $y = 0$. Hence (4.28) becomes

$$\frac{v_0^2}{2} = gR + c$$

or

$$c = \frac{v_0^2}{2} - gR.$$

Substituting this (4.28) gives,

$$v = \pm\sqrt{v_0^2 - 2gR + \frac{2gR^2}{R+y}} \tag{4.29}$$

Note that Eq.(4.29) gives the velocity $v$ as a function of $y$, the altitude, rather than as a function of time. The plus sign must be chosen if the body is rising, and the minus sign if it is falling back to earth.

**(b)** To determine the maximum altitude $\xi$ that the body reaches, we note that at the maximum altitude, velocity is zero. Hence we set $v = 0$ and $y = \xi$ in Eq.(4.29) and then solve for $\xi$, obtaining

$$\xi = \frac{v_0^2 R}{2gR - v_0^2} \tag{4.30}$$

Solving Eq.(4.30) for $v_0$, we find the initial velocity required to lift the body to the maximum altitude $\xi$, we obtain

$$v_0 = \sqrt{2gR\frac{\xi}{R+\xi}} \tag{4.31}$$

**(c)** The escape velocity $v_e$ is found by letting $\xi \to \infty$ in (4.31). That is,

$$\begin{aligned} v_e &= \sqrt{2gR\lim_{\xi\to\infty}\frac{\xi}{R+\xi}} = \sqrt{2g\,R\lim_{\xi\to\infty}\frac{1}{\frac{R}{\xi}+1}} \\ &= \sqrt{2gR}. \end{aligned} \tag{4.32}$$

***Example* 7** (*Separable and Linear First Order Differential Equations*) Suppose that a sum of money is deposited in a bank that pays interest at an annual rate $r$. The value $S(t)$ of the investment at any time $t$ depends on the frequency with which interest is compounded as well as on the interest rate. Banks have various policies concerning compounding: some compound monthly, some weekly, some even daily.

(a) Assuming that compounding takes place *continuously,* set up a simple initial value problem that describes the growth of the investment. Then solve the initial value problem.

(b) Compare the result in the continuous model (a) with the situation in which compounding occurs at finite time intervals.

(c) In the case of continuous compounding, also assume that there may be deposits or withdrawals in addition to the accrual of interest. Set up and solve the initial value problem.

*Solution*

$\frac{dS}{dt}$, the rate of change of the value of the investment, is equal to the rate at which interest accrues, which is the interest rate $r$ times the current value of the investment $S(t)$. Thus

$$\frac{dS(t)}{dt} = rS(t),$$

or simply

$$\frac{dS}{dt} = rS \tag{4.33}$$

is the differential equation that governs the process. Suppose the value of the investment at initial time is, $S_0$. Then

$$S(0) = S_0. \tag{4.34}$$

Then the solution of the initial value problem given by (4.33) and (4.34) gives the balance $S(t)$ in the account at any time $t$. This initial value problem is readily solved, since the differential equation (4.33) is both linear and separable.

Eqn. (4.33) can be written in the separable form

$$\frac{dS}{S} = r\,dt.$$

Integrating,

$$\ln\ S = r\,t + C.$$

Using the initial condition $S = S_0$ at $t = 0$, we have

$$\ln\ S_0 = C$$

Hence general solution is

$$\ln\ S = rt + \ln\ S_0$$

or

$$\ln\frac{S}{S_0} = rt.$$

or

$$S(t) = S_0 e^{rt} \tag{4.35}$$

**Remark** Thus a bank account with continuously compounding interest grows exponentially.

**(b)** If interest is compounded once a year, then after $t$ years

$$S(t) = S_0(1 + r)^t.$$

If interest is compounded twice a year, then at the end of 6 months the value of the investment is $S_0[1+(r/2)]$, and at the end of 1 year it is $S_0[1 + (r/2)]^2$. Thus, after $t$ years we have

$$S(t) = S_0 \left(1 + \frac{r}{2}\right)^{2t}$$

In general, if interest is compounded $m$ times per year, then

$$S(t) = S_0 \left(1 + \frac{r}{m}\right)^{mt}. \tag{4.36}$$

**Remark** We recall from calculus that

$$\lim_{m\to\infty} \left(1 + \frac{r}{m}\right)^{mt} = e^{rt},$$

and using this, we have

$$\lim_{m\to\infty} S_0 \left(1 + \frac{r}{m}\right)^{mt} = S_0 e^{rt},$$

and hence the relation between formulas (4.35) and (4.36) is justified.

**(c)** In the case of continuous compounding, let us suppose that there may be deposits or withdrawals in addition to the accrual of interest. If we assume that the deposits or withdrawals take place at a constant rate $k$, then Eq. (4.33) is replaced by

$$\frac{dS}{dt} = rS + k,$$

or, in standard form,

$$\frac{dS}{dt} - rS = k, \tag{4.37}$$

where $k$ is positive for deposits and negative for withdrawals. (4.37) is a first order linear equation. Its integrating factor is

$$\mu(t) = e^{\int -r dt} = e^{-rt}.$$

Hence,

$$\frac{d}{dt}(\mu(t)\, S(t)) = k\mu(t)$$

Integrating,

$$(\mu(t)\, S(t)) = \int k\mu(t)dt + c$$

i.e.,

$$e^{-rt}S(t) = k \int e^{-rt}dt + c$$

i.e.,

$$e^{-rt}\, S(t) = \frac{-k}{r} \cdot e^{-rt} + c.$$

Hence general solution is

$$S(t) = ce^{rt} - (k/r),$$

where $c$ is an arbitrary constant. To satisfy the initial condition (4.34),

$$S_0 = S(0) = C - \frac{k}{r}$$

which gives

$$c = S_0 + (k/r).$$

Thus the solution of the initial value problem (4.37), with (4.34) is

$$S(t) = S_0 e^{rt} + (k/r)(e^{rt} - 1). \tag{4.38}$$

The first term in expression (4.38) is the part of $S(t)$ that is due to the return accumulated on the initial amount $S_0$, and the second term is the part that is due to the deposit or withdrawal rate $k$.

***Example* 8** Suppose that one opens an individual retirement account (IRA) at age 25 and makes annual investments of Rs.2000 thereafter in a continuous manner. Assuming a rate of return of 8%, what will be the balance in the IRA at age 65?

*Solution.*

We have

$S_0 = 0$, $r = 0.08$, and $k =$ Rs 2000,

and we wish to determine $S(40)$. From Eq. (4.38) we have ( with $t = 40$)

$$S(40) = \frac{2000}{0.08}(e^{0.08\times 40} - 1) = (25,000)(e^{3.2} - 1) = \text{Rs } 588,313. \tag{4.39}$$

**Remark** It is interesting to note that the total amount invested is $40 \times 2000 =$ Rs. 80,000, so the remaining amount of Rs.$508,313$ results from the accumulated return on the investment.

***Example* 9** Suppose that $x_0$ bacteria are placed in a nutrient solution at time $t = 0$, and that $x = x(t)$ is the population of the colony at a later time $t$. If food and living space are unlimited, and if as a consequence the population at any moment is increasing at a rate proportional to the population at that moment, find $x$ as a function of $t$.

*Solution*

Since the rate of increase of $x$ is proportional to $x$ itself, the differential equation is

$$\frac{dx}{dt} = kx,$$

where $k$ is a proportionality constant.

By separating the variables,

$$\frac{dx}{x} = k\,dt,$$

Integrating,

$$\ln x = kt + c.$$

Since $x = x_0$ when $t = 0$, we have

$$c = \ln x_0$$

so

$$\ln x = kt + \ln x_0$$

and

$$x = x_0 e^{kt}. \tag{4.40}$$

***Example* 10** Suppose the human population in the earth is increasing at an overall rate of approximately 2 percent per year of the population at that time. Find the 'doubling time' $T$, that is, the time needed for the total number of people in the world to increase by a factor of 2.

*Solution*

Here the population at any moment is increasing at a rate proportional to the population at that moment, and it is given that proportionality constant is 2 percent. i.e., $k = \frac{2}{100} = 0.02 = \frac{1}{50}$. Hence proceeding as in the previous example, we obtain

$$x = x_0 e^{t/50} \tag{4.41}$$

To find the *doubling time* $T$, that is, the time needed for the total number of people in the world to increase by a factor of 2, we replace (4.41) by

$$2x_0 = x_0 e^{T/50}.$$

This yields

$$T/50 = \ln 2,$$

so

$T = 50 \ln 2 \cong 34.65$ years,

since natural logarithm of 2,$\ln 2 \cong 0.693$.

***Example* 11** A tank contains 50 gallons[2] of brine (water with a high salt content) in which 75 pounds[3] of salt are dissolved. Beginning at time $t = 0$, brine containing 3 pounds of salt per gallon flows in at the rate of 2 gallon per minute, and the mixture (which is kept uniform by stirring) flows out at the same rate. When will there be 125 pounds of dissolved salt in the tank? How much dissolved salt is in the tank at time after a long time?

*Solution*

If $x = x(t)$ is the number of pounds of dissolved salt in the tank at time $t \geq 0$, then the concentration at that time is $x/50$ pounds per gallon.

The rate of change of $x$ is

$\frac{dx}{dt}$ =rate at which salt enters tank – rate at which salt leaves tank.

Since

rate of entering $= \ 3 \cdot 2 = 6 \, \mathrm{lb}/\min$

and

rate of leaving $= \ \left(\frac{x}{50}\right) \cdot 2 = \frac{x}{25} \ \mathrm{lb}/\min,$

we have

$$\frac{dx}{dt} = 6 - \frac{x}{25} = \frac{150 - x}{25}.$$

Separating variables,

$$\frac{dx}{150 - x} = \frac{1}{25} dt.$$

[2]Gallon is a unit of volume for liquid measure equal to 4.55 litres.

[3]Pounds is a unit of weight.

Integrating,

$$\ln(150 - x) = -\frac{1}{25}t + c.$$

Since $x = 75$ when $t = 0$, we see that $c = \ln 75$, so

$$\ln(150 - x) = -\frac{1}{25}t + \ln 75, ,$$

and therefore

$$150 - x = 75e^{-t/25}$$

or

$$x = 75(2 - e^{-t/25}).$$

Hence $x = 125$ implies $e^{t/25} = 3$ or $t/25 = \ln 3$. Thus $x = 125$ pounds after $t = 25\ \ln 3 \cong 27.47$ minutes.

Also, when $t$ is large we see that $x$ is nearly $75 \cdot 2 = 150$ pounds, as common sense tells us without calculation.

### 4.1.1 Exercises Set A

In Exercises 1-15, solve the differential equations.

1. $y' = ky$
2. $y' - 2y + a = 0$
3. $y' = -xy$
4. $xy' + by = 0$
5. $(x \log x)y' = y$
6. $\frac{dy}{dx} - \frac{2}{x}\sqrt{y-1} = 0$
7. $(x+2)\frac{dy}{dx} = xy$
8. $yy' - 2xe^{y^2} = 0$
9. $2\frac{dy}{dx} = y \cot x$
10. $y' = (1+x)(1+y^2)$

11. $\frac{dy}{dx} = -\csc y$

12. $yy' - 0.5\sin^2 ax = 0$

13. $\frac{dy}{dx} - y\tanh x = 0$

14. $y' - y\cot 2x = 0$

15. $(1+x^2)\frac{dy}{dx} = 1 + y^2$

In Exercises 16-23, solve the initial value problems.

16. $\frac{dy}{dx} - y\tan 2x = 0; \quad y(0) = 2$

17. $2xy' - 3y = 0; \quad y(1) = 4$

18. $(x+1)y' - 2y = 0; \quad y(0) = 1$

19. $xy'\log x - y = 0; \quad y(2) = \log 4$

20. $\frac{dy}{dx} = 2e^x y^3; \quad y(0) = 0.5$

21. $\frac{dr}{dt} + tr = 0; \quad r(0) = r_0$

22. $dr\ \sin\theta - 2r\cos\theta\ d\theta = 0; \quad r(\pi/2) = 2$

23. $(x^2+1)\frac{dy}{dx} + y^2 + 1 = 0; \quad y(0) = 1$

24. $y' = \frac{xy^3}{\sqrt{1+x^2}}, \quad y(0) = 1$

25. $y' = \sec y; \quad y(0) = 0$

## Answers

1. $y = ce^{kx}$

2. $y = ce^{2x} + \frac{1}{2}a$

3. $y = ce^{-\frac{x^2}{2}}$

4. $yx^b = c$

5. $y = c \log x$
6. $y = (\log x + c)^2 + 1$
7. $(x+2)^2 y = ce^x$
8. $e^{-y^2} + 2x^2 = c$
9. $y = c(\sin x)^{1/2}$
10. $y = \tan(\frac{1}{2}x^2 + x + c)$
11. $x - \cos y = c$
12. $4y^2 = 2x - \frac{1}{a}\sin 2ax + c$
13. $y = c \cosh x$
14. $y^2 = \sin 2x + c$
15. $\tan^{-1} y = \tan^{-1} x + c$
16. $y^2 \cos 2x = 4$
17. $y = 4x^{3/2}$
18. $y = (1+x)^2$
19. $y = \log(2+x)$
20. $y^2 = (8 - 4e^x)^{-1}$
21. $r = r_0 e^{\frac{-t^2}{2}}$
22. $r = 2\sin^2\theta$
23. $y = \frac{1-x}{1+x}$
24. $\frac{1}{y^2} = 3 - 2\sqrt{1+x^2}$
25. $y = \sin^{-1} x$

**Exercises Set B**

In Exercises 1-11, solve the given differential equation.

1. $\frac{dy}{dx} + \frac{x}{y} = 0$
2. $\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0$
3. $(1+x)y dx + (1-y)x dy = 0$
4. $(1+x^2)dy + x\sqrt{1-y^2}dx = 0$
5. $(x^2 - yx^2)dy + (y^2 + xy^2)y dx = 0$

6. $(e^x + 1)ydy = e^x(y + 1)dx$

7. $\sqrt{1 + x^2}dy + \sqrt{1 + y^2}dx = 0$

8. $\sec^2 x \tan y dx + \sec^2 y \tan x dy = 0$

9. $ye^{2x}dx - (1 + e^{2x})dy = 0$

10. $3e^x \tan y dx + (1 - e^x) \sec^{2y} dy = 0$

11. Solve $\frac{dy}{dx} = \frac{x - e^{-x}}{y + e^y}$

    In Exercises 12-17: (a) Find the solution of the given initial value problem in explicit form. (b) Determine (at least approximately) the interval in which the solution is defined.

12. $y' = (1 - 2x)y^2, \quad y(0) = -1/6$

13. $xdx + ye^{-x}dy = 0, \quad y(0) = 1$

14. $y' = 2x/(y + x^2y), \ \ y(0) = -2$

15. $y' = 2x/(1 + 2y), \quad y(2) = 0$

16. $y' = (3x^2 - e^x)/(2y - 5), \ \ y(0) = 1$

17. $\sin 2x dx + \cos 3y dy = 0, \quad y(\pi/2) = \pi/3$

18. (Newton's law of cooling). Assume that the rate at which a hot body cools is proportional to the difference in temperature between it and its surroundings. A body is heated to $110^0\,C$ and placed in air at $10^0\,C$. After I hour its temperature is $60^0\,C$. How much additional time is required for it to cool to $30^0\,C$?

19. A pot of carrot and garlic soup cooling in air at $0^0\,C$ was initially boiling at $100^0\,C$ and cooled $20^0\,C$ during the first 30 minutes. How much will it cool during the next 30 minutes?

20. The radiocarbon in living wood decays at the rate of 15.30 disintegrations per minute (dpm) per gram of contained carbon. Using 5600 years as the half-life of radiocarbon, estimate the age of each of the following specimens discovered by archaeologists and tested for radioactivity:

    a) a piece of a chair leg from the tomb of King Tutankhamen, 10.14 dpm;

    b) a piece of a beam of a house built in Babylon during the reign of king Hammurabi, 9.52 dpm;

    c) dung of a giant sloth found 6 feet 4 inches under the surface of the ground inside Gypsum Cave in Nevada, 4.17 dpm;

    d) a hardwood spear-thrower found in Leonard Rock shelter in Nevada, 6.42 dpm.

***Answers***

1. $x^2 + y^2 = a^2$
2. $\sin^{-1} x + \sin^{-1} y =$c
3. $x - y + \log xy = c$
4. $\sin^{-1} y + \frac{1}{2}\log(1 + x^2) = c$
5. $\frac{1}{x} - \frac{1}{y} + \frac{1}{2y^2} = c + \log x$
6. $e^y = c(y + 1)(e^x + 1)$
7. $\sinh^{-1} x + \sinh^{-1} y = c$
8. $\tan x \tan y = c$

9. $1 + e^{2x} = cy^2$

10. $\tan y = c(1 - e^x)^3$

11. $y^2 - x^2 + 2(e^y - e^{-x}) = c, \quad y + e^y \neq 0$

12. (a) $y = 1/(x^2 - x - 6)$ (b) $-2 < x < 3$

13. (*a*) $y = [2(1 - x)e^x - 1]^{1/2}$ (b) $-1.68 < x < 0.77$

14. (*a*) $y = -[2\ln(1 + x^2) + 4]^{1/2}$ (b) $-\infty < x < \infty$

15. (*a*) $y = -\frac{1}{2} + \frac{1}{2}\sqrt{4x^2 - 15}$ (b) $x > \frac{1}{2}\sqrt{15}$

16. (*a*) $y = \frac{5}{2} - \sqrt{x^3 - e^x + \frac{13}{4}}$ (b) $-1.4445 < x < 4.6297$

17. (*a*) $y = [\pi - \arcsin(3\cos^2 x)]/3$ (b) $|x - \pi/2| < 0.6155$

18. $\frac{\ln 5}{\ln 2} - 1$ hours

19. $16^0\, C$

20. (a) about 3330 years (b) about 3850 years

    (c) about 10,510 years (d) about 7010 years

## 4.2 Equations Reducible to Separable Form: By Substitution

Certain first order differential equations are not separable but can be made separable by simple substitutions, which is illustrated through the following examples.

***Example* 12** Solve the differential equation

$$(2x - 4y + 5)y' + x - 2y + 3 = 0.$$

*Solution*

Put $x - 2y = u$.

Then, differentiating both sides with respect to $x$, we obtain

$$1 - 2\frac{dy}{dx} = \frac{du}{dx}$$

or

$$\frac{dy}{dx} = \frac{1}{2}\left(1 - \frac{du}{dx}\right)$$

Substituting these values in the given differential equation, we obtain

$$(2u + 5)\frac{1}{2}\left(1 - \frac{du}{dx}\right) + u + 3 = 0$$

or

$$(2u + 5)\frac{du}{dx} = 4u + 11.$$

Separating variables, we get

$$\left(\frac{2u + 5}{4u + 11}\right) du = dx,$$

which, by actual division,[4] gives

$$\frac{1}{2}\left(1-\frac{1}{4u+11}\right)du = dx.$$

On integration, we get

$$\frac{1}{2}\left(u-\frac{1}{4}\log|4u+11|\right) = x+c$$

or

$$u-\frac{1}{4}\log|4u+11| = 2x+C,$$

where $C$=2$c$.

Putting $u = x - 2y$, the above equation becomes

$4x + 8y + \log 4x\text{-}8y\text{+}11\text{=} C.$

## 4.3 Exercises

In Exercises 1-8, using appropriate substitutions, find the general solution of the following differential equations.

1. $y' = (y-x)^2$
2. $y' = \tan(x+y) - 1$
3. $xy' = e^{-xy} - y.$
4. $y' = (x+e^y-1)e^{-y}$
5. $y' = \frac{y-x+1}{y-x+5}$
6. $\frac{dy}{dx} = \frac{1-2y-4x}{1+y+2x}$
7. $(2x-4y+5)dy = (x-2y+3)dx$

---

[4] Actual division gives, $\frac{2u+5}{4u+11} = \frac{1}{2}\left(1-\frac{1}{4u+11}\right)$

8. $\frac{dy}{dx} = \frac{y}{x}\left(\log\frac{y}{x} + 1\right)$

9. Solve the initial value problem:

   $2x^2y\frac{dy}{dx} = \tan(x^2y^2) - 2xy^2;\ y(1) = \sqrt{\pi/2}$

* Hints: Substitution required in the exercises are:

1. $y - x = u$
2. $x + y = u$
3. $xy = u$
4. $x + e^y = u$
5. $y - x = u$
6. $y + 2x = u$
7. $x - 2y = u$
8. $y/x = u$
9. $x^2y^2 = u.$

**Answers**

1. $y = x + \frac{1+ce^{2x}}{1-ce^{2x}}$
2. $\sin(x+y) = ce^x$
3. $xy = \log(x+c)$
4. $x + e^y = ce^x$
5. $(y-x)^2 + 10y - 2x = c$
6. $(y+2x)^2 + 2(y-x) = c$
7. $(x-2y)^2 + 5(x-2y) = -x + c.$
8. $y = xe^{cx}$ or $\log\frac{y}{x} = cx$
9. $\log\sin(x^2y^2) = x - 1$

## 4.4 Equations Reducible to Separable Form: Homogeneous Equation $\frac{dy}{dx} = g\left(\frac{y}{x}\right)$

If the right side of the equation

$$\frac{dy}{dx} = f(x,\, y)$$

can be expressed as a function of the ratio $\frac{y}{x}$ only, then the equation is said to be homogeneous. More precisely, a differential equation of the form

$$\frac{dy}{dx} = g\left(\frac{y}{x}\right) \tag{4.42}$$

where $g$ is any given function of $\frac{y}{x}$ [for example $\left(\frac{y}{x}\right)^2$, $\cos\left(\frac{y}{x}\right)$, $1+\frac{y}{x}$, etc.] is called a homogeneous equation.

The form of equation (4.42) suggests that we may introduce a new variable $u$ such that

$$u = \frac{y}{x},$$

where $y$ and $u$ are functions of $x$. Then

$$y = xu.$$

Differentiating with respect to $x$, we get

$$\frac{dy}{dx} = u + x\frac{du}{dx} \tag{4.43}$$

By inserting this into (4.42) and noting that $g\left(\frac{y}{x}\right) = g(u)$, we obtain

$$u + x\frac{du}{dx} = g(u).$$

Separating the variables $u$ and $x$, we have the separable form

$$\frac{du}{g(u) - u} = \frac{dx}{x} \tag{4.44}$$

If we integrate (4.44) and then replace $u$ by $y/x$, we obtain the general solution of (4.42).

***Example* 13** Solve the first order differential equation

$$2xy\frac{dy}{dx} - y^2 + x^2 = 0.$$

*Solution*

The given equation can be written as

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy} = \frac{\left(\frac{y}{x}\right)^2 - 1}{2\left(\frac{y}{x}\right)}, \tag{4.45}$$

and hence is a homogenous equation.

Put $u = \frac{y}{x}$. Then $y = ux$, which on differentiation with respect to $x$ yields

$$\frac{dy}{dx} = u + x\frac{du}{dx}.$$

Substitution on the differential equation (4.45) gives

$$u + x\frac{du}{dx} = \frac{u^2 - 1}{2u}$$

implies

$$x\frac{du}{dx} = \frac{u^2 - 1}{2u} - u = -\frac{u^2 - 1}{2u}..$$

Separating the variables, we obtain

$$\frac{2u\,du}{1 + u^2} = -\frac{dx}{x}.$$

On integration, we obtain

$$\ln\left|1+u^2\right| = -\ln x + \ln c,$$

where $c$ is an arbitrary constant. Hence

$$1+u^2 = \frac{1}{cx}.$$

Replacing $u$ by $y/x$, we obtain

$$1+\left(\frac{y}{x}\right)^2 = \frac{1}{cx}$$

or

$$\frac{x^2+y^2}{x^2} = \frac{1}{cx}$$

or

$$x^2+y^2 = \frac{1}{c}\cdot x$$

i.e.,

$$x^2+y^2 = Cx,$$

where $C = \frac{1}{c}$, is the general solution to the given differential equation.

**Exercises**

1. $x(x-y)\frac{dy}{dx} + y^2 = 0$
2. $\frac{dy}{dx} = -\frac{x^2+3y^2}{3x^2+y^2}$
3. $\frac{dy}{dx} = \frac{x^2-4xy-2y^2}{2x^2+4xy-y^2}$
4. $\frac{dy}{dx} = \frac{x^2+y^2}{x^2+xy}$
5. $x(y-x)dy = (y+x)ydx$

6. $(xy - 2y^2)dx - (x^2 - 3xy)dy = 0$

7. $\frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x}$

8. $(x^2 - 2xy - y^2)dx - (x + y)^2 dy = 0$

9. $(x^3 - 3xy^2)dy = (y^3 + 3x^2 y)dx$

10. $xy^2 \frac{dy}{dx} = x^3 + y^3$

***Answers***

1. $y = ce^{\frac{y}{x}}$.

2. $\frac{2xy}{(x+y)^2} + \log(x + y) = c$

3. $x^3 - 6x^2 y - 6xy^2 - y^3 = c$

4. $(x - y)^2 e^{\frac{y}{x}} = cx$

5. $xy = ce^{\frac{y}{x}}$

6. $y^3 = cx^2 e^{-\frac{x}{y}}$

7. $y + \sqrt{x^2 + y^2} = cx^2$

8. $c + x^3 = y^3 + 3x^2 y + 3xy^2$

9. $(x^2 - y^2)^2 = cxy$

10. $y^3 = 3x^3(\ln x + c)$

## 4.5 Equations Reducible to Separable Form: via Reducing to Homogeneous Equation

Certain equation may not be in the homogeneous form, but can be made to homogeneous form by appropriate change of variables. This is illustrated in the following example.

***Example* 14** Solve $\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$.

*Solution*

*The given equation is not readily homogeneous. But it can be reduced to the homogeneous type by substituting*

$x = X + h$ and $y = Y + k$

where $h$ and $k$ are constants to be determined later. From the substitution, we have

$$\frac{dy}{dx} = \frac{dY}{dX}$$

and the equation takes the form

$$\frac{dY}{dX} = \frac{(X + 2Y) + (h + 2k - 3)}{(2X + Y) + ((2h + k - 3)} \tag{4.46}$$

Now, $h$ and $k$ are chosen so that $h+2k-3 = 0$ and $2h+k-3 = 0$, solving which we get $h = 1$ and $k = 1$. Therefore (4.46) takes the form

$$\frac{dY}{dX} = \frac{X + 2Y}{2X + Y} \tag{4.47}$$

which is a homogeneous equation. By substituting

$$Y = VX \text{ and } \frac{dY}{dX} = V + X\frac{dV}{dX},$$

equation (4.47) takes the form
$X\frac{dV}{dX} = \frac{1+2V}{2+V} - V$ or $\frac{dX}{X} = \frac{2+V}{1-V^2}dV$ or $\frac{dX}{X} = \left[\frac{1/2}{1+V} + \frac{3/2}{1-v}\right] dV$
Integration of the above yields

$$\log X = \frac{1}{2}\log(1+V) - \frac{3}{2}\log(1-V) + \log C,$$

where $C$ is an arbitrary constant. The above can be simplified to

$$X^2(1-V)^3 = C^2(1+V).$$

Substituting the values of $X$ and $V$, we get the required general solution as

$$(x-y)^3 = C^2(x+y-2).$$

**Exercises**

1. $\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$
2. $\frac{dy}{dx} = \frac{3x-5y-9}{2x-4y-8}$
3. $\frac{dy}{dx} = \frac{2x+9y-20}{6x+2y-10}$
4. $\frac{dy}{dx} = \frac{x+7y+2}{3x+5y+6}$
5. $\frac{dy}{dx} = \frac{x+7y+2}{3x+5y+6}$
6. .$(x-y-2)dx + (x+y)dy = 0$
7. $(3x+y-5)dy = 2(x+y-1)dx$
8. $\frac{dy}{dx} = \frac{2x+3y+4}{4x+6y+5}$
9. $(2x-4y+3)\frac{dy}{dx} + (x-2y+1) = 0$
10. $\frac{dy}{dx} = \frac{2x+y-3}{4x+2y+7}$

***Answers***

1. $x+y-2 = c(x-y)^3$

2. $(y - x + 1)^2 = c(4y - 3x + 6)$

3. $(y - 2x)^2 = c(2y + x - 5)$

4. $(x - y + 2)^4 = c(x + 5y + 2)$

5. $\log(x - y + 2) = 2y - x + c$

6. $\log[(x + 1)^2 + (y - 1)^2] + 2\tan^{-1}\left[\frac{y-1}{x+1}\right] = c$

7. $(x - y - 3)^4 = c(2x + y - 3)$

8. $9\log(14x + 21y + 22) + 21x - 42y + c = 0$

9. $\log(4x - 8y + 5) = 4x + 8y + c$

10. $13\log(10x + 5y + 11) = 5x - 10y - c.$

# Chapter 5

# Differences Between Linear and Nonlinear Equations

## 5.1 Linear and Nonlinear Equations

The first theorem in this chapter says that if the underlying differential equation of an initial value problem is linear, then the solution to the IVP is unique. We will see that this need not be true for initial value problems with non-linear differential equations.

**Theorem 1 [Existence and Uniqueness Theorem for First-Order Linear Equations]** If the functions $p$ and $g$ are continuous on an open interval $I : \alpha < t < \beta$ containing the point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies the linear differential equation

$$y' + p(t)y = g(t) \tag{5.1}$$

for each $t$ in $I$, and that also satisfies the initial condition

$$y(t_0) = y_0, \tag{5.2}$$

where $y_0$ is an arbitrary prescribed initial value.

***Example* 1** Use Theorem 1 to find an interval in which the initial value problem

$$ty' + 2y = 4t^2 \tag{5.3}$$

$$y(1) = 2 \tag{5.4}$$

has a unique solution. Also find the solution.

*Solution*

Eq.(5.3) can be brought into the standard form (5.1) as follows:

$$y' + \frac{2}{t}y = 4t$$

Hence $p(t) = 2/t$ and $g(t) = 4t$. Thus for this equation, $g$ is continuous for all $t$, while $p$ is continuous only for $t < 0$or for $t > 0$. The interval $t > 0$ contains the initial point $t = 1$; consequently, Theorem 1 guarantees that the initial value problem (5.3) with (5.4) has a unique solution on the interval $0 < t < \infty$.

Here integrating factor is

$$\mu(t) = e^{\int p(t)\,dt} = e^{\int \frac{2}{t}dt} = e^{2\ln t} = t^2$$

$$\therefore \frac{d}{dt}(\mu(t)y) = 4\mu(t)t.$$

Integrating,

$$\mu(t)y = \int 4t^3 dt + C$$

$$t^2 y = t^4 + C$$

or

$$y = t^2 + \frac{C}{t^2}.$$

The initial condition $y = 2$ when $t = 1$ gives $2 = 1 + C$ or $C = 1$. Hence the solution of this initial value problem is

$$y = t^2 + \frac{1}{t^2}, \qquad t > 0 \tag{5.5}$$

***Example* 2** Solve the initial value problem

$$ty' + 2y = 4t^2 \tag{5.6}$$

$$y(-1) = 2 \tag{5.7}$$

*Solution*

Comparing with the IVP in Example 1, here only the initial condition (5.4) is changed to $y(-1) = 2$. Theorem 1 asserts the

existence of a unique solution for $t < 0$. Proceeding as in Example 1, the solution is given by $y = t^2 + \frac{1}{t^2}, \qquad t < 0.$
The following is a general version of Theorem 1.

**Theorem 2 [Existence and Uniqueness Theorem for First-Order Nonlinear Equations]** Let the functions $f$ and $\partial f/\partial y$ be continuous in some rectangle (Fig.5.1) $\alpha < t < \beta$, $\gamma < y < \delta$containing the point $(t_0, y_0)$. Then, in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = \phi(t)$ of the initial value problem

$$y' = f(t, y), \qquad y(t_0) = y_0. \tag{5.8}$$

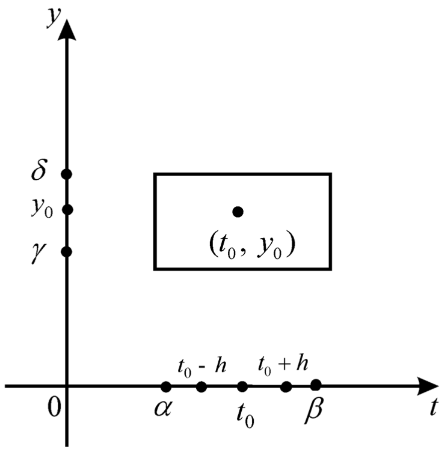


Figure 5.1:

**Remark** If all the conditions of Theorem 2 except the continuity of $\frac{\partial f}{\partial y}$ are satisfied, then the solution has a solution (but not unique).

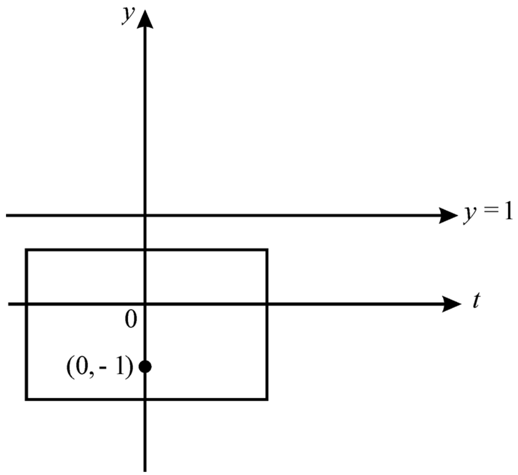


Figure 5.2:

***Example* 3** Apply Theorem 2 to the initial value problem

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}, \qquad y(0) = -1 \tag{5.9}$$

*Solution*

Here

$$f(x, y) = \frac{3x^2 + 4x + 2}{2(y-1)},$$

and

$$\frac{\partial f}{\partial y} = \frac{3x^2 + 4x + 2}{2} \frac{\partial}{\partial y}\left(\frac{1}{y-1}\right) = -\frac{3x^2 + 4x + 2}{2(y-1)^2}.$$

Each of these functions is continuous everywhere except on the line $y = 1$. Hence, a rectangle (Fig. 5.2) can be drawn about the initial point $(0, -1)$ in which both $f$ and $\partial f/\partial y$ are continuous. Hence, Theorem 2 guarantees that the initial value problem has a unique solution in some interval about $x = 0$. It has been seen in Example 5 of the previous chapter that the solution to the initial value problem (5.9) is

$$y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}.$$

Also, the above solution is valid when $x^3 + 2x^2 + 2x + 4 > 0$. The only real zero of $x^3 + 2x^2 + 2x + 4 = 0$ is $x = -2$, and hence the solution is valid only when $x > -2$.

**Attention!** We note that even though the rectangle can be stretched infinitely far in both the positive and negative $x$-directions, this does not necessarily mean that the solution exists for all $x$. Indeed, the solution of the initial value problem (5.9) exists only for $x > -2$.

[**Attention!** Theorem 1 is not applicable to the initial value problem (5.9) in Example 3, since the differential equation is **nonlinear.**]

***Example* 4** (*IVP with more than one solution*) Can we apply Theorem 2 to the IVP

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}, \quad y(0) = 1 \ ?$$

Also solve the IVP.

*Solution*

The initial point (0, 1) lies on the line $y = 1$. Since $f$ and $\frac{\partial f}{\partial y}$ are not defined at $y = 1$, no rectangle can be drawn about (0, 1), within which $f$ and $\partial f/\partial y$ are continuous. Hence, Theorem 2 says nothing about possible solutions of the given initial value problem. Separating the variables we obtain

$$2(y-1)dy = (3x^2 + 4x + 2)dx.$$

Integrating, we obtain

$$y^2 - 2y = x^3 + 2x^2 + 2x + c.$$

Further, if $x = 0$ and $y = 1$, then $c = -1$. Hence

$$y^2 - 2y = x^3 + 2x^2 + 2x - 1.$$

Solving for $y$, the above quadratic equation in $y$ gives

$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x} \tag{5.10}$$

**Remark** The above example deals a case where Theorem 1 and Theorem 2 cannot be applied. Also note that this IVP has more than one solution. Equation (5.10) provides two functions that satisfy the given differential equation for $x > 0$ and also satisfy the initial condition $y(0) = 1$.

***Example* 5** (*IVP with infinite number of solutions*) Can we apply

Theorem 2 to the initial value problem

$$\frac{dy}{dt} = y^{1/3}, \qquad y(0) = 0 \tag{5.11}$$

for $t \geq 0$? Solve the initial value problem.

*Solution*

The function $f(t, y) = y^{1/3}$ is continuous everywhere, but $\partial f/\partial y$ does not exist when $y$=0, and hence is not continuous there. Thus Theorem 2 does not apply to this problem and no conclusion can be drawn from it. However, by the remark following Theorem 2, the continuity of $f$ does ensure the existence of solutions, but not their uniqueness.

Obviously, the function

$$y = \psi(t) = 0, \quad t \geq 0 \tag{5.12}$$

is a solution to the IVP (5.11).

To find other solutions of (5.11), we solve the differential equation. By separating variables,

$$y^{-1/3} dy = dt.$$

Integrating,

$$\frac{3}{2} y^{2/3} = t + c.$$

Hence

$$y = \left[\frac{2}{3}(t + c)\right]^{3/2}$$

The initial condition $y = 0$ when $t = 0$ gives $c = 0$. Hence a solution to the IVP is

$$y = \phi_1(t) = \left(\frac{2}{3}t\right)^{3/2}, \quad t \geq 0 \tag{5.13}$$

Also the function

$$y = \phi_2(t) = -\left(\frac{2}{3}t\right)^{3/2}, \; t \geq 0 \tag{5.14}$$

is also a solution of the initial value problem. Also, for an arbitrary positive $t_0$, the functions

$$y = \chi(t) = \begin{cases} 0 & \text{if } \; 0 < t < t_0 \\ \pm\left[\frac{2}{3}(t - t_0)\right]^{3/2} & \text{if } \; t \geq t_0 \end{cases} \tag{5.15}$$

are continuous, differentiable (in particular at $t = t_0$), and are solutions of the initial value problem (5.15).

We conclude that the given IVP has an infinite number of solutions.

**Remark to Example 5**

1. As noted in the solution of Example 5, the existence and uniqueness theorem is not applicable if $y = 0$. i.e., when the initial point (0, 0) lies on the $t$-axis. If the initial point $(t_0, y_0)$is any point not on the $t$-axis, however, then the theorem guarantees that there is a unique solution of the differential equation $y' = y^{1/3}$ passing through $(t_0, y_0)$.

**Interval of Definition or Interval of Existence**

According to Theorem 1, the solution of a linear equation

$$y' + p(t)y = g(t),$$

subject to the initial condition $y(t_0) = y_0$, exists throughout any interval about $t = t_0$ in which the functions $p$ and $g$ are continuous. Thus, vertical asymptotes or other discontinuities in the solution can occur only at points of discontinuity of $p$ or $g$. For instance, the solution in Example 1 (with one exception) are asymptotic to the $y$-axis, corresponding to the discontinuity at $t = 0$in the coefficient $p(t) = 2/t$, but none of the solutions has any other point where it fails to exist and to be differentiable. The one exceptional solution shows that solutions may sometimes remain continuous even at point of discontinuity of the coefficients.

On the other hand, for a nonlinear initial value problem satisfying the hypotheses of Theorem 2, the interval in which a solution exists may be difficult to determine. The solution $y = \phi(t)$is certain to exist as long as the point $[t, \phi(t)]$ remains within a region in which the hypotheses of Theorem 2 are satisfied. This is what determines the value of $h$ in that theorem. However, since $\phi(t)$ is usually not known, it may be impossible to locate the point $[t, \phi(t)]$ with respect to this region. In any case, the interval in which a solution exists may have no simple relationship to the function $f$ in the differential equation $y' = f(t, y)$. This is illustrated by the following example.

***Example* 6** Solve the initial value problem

$$y' = y^2, \qquad y(0) = 1 \tag{5.16}$$

and determine the interval in which the solution exists.

*Solution*

Since $f(t, y) = y^2$ and $\frac{\partial f}{\partial y} = 2y$ are continuous everywhere, by Theorem 2, the given initial value problem has a unique solution. To find the solution, we separate the variables to obtain

$$\frac{dy}{y^2} = dt$$

i.e.,

$$y^{-2}dy = dt. \tag{5.17}$$

Integrating,

$$-y^{-1} = t + c.$$

Then, solving for $y$, we have

$$y = -\frac{1}{t+c} \tag{5.18}$$

The initial condition $y(0) = 1$ gives $c = -1$, so

$$y = \frac{1}{1-t} \tag{5.19}$$

is the solution of the given initial value problem. Clearly, the solution becomes unbounded as $t \to 1$; therefore, the solution exists only in the interval $-\infty < t < 1$.

***Example* 7** Solve the initial value problem

$$y' = y^2, \tag{5.20}$$

with the initial condition

$$y(0) = y_0, \tag{5.21}$$

and determine the interval in which the solution exists.

*Solution*

The only difference of the given IVP that in the previous example is that the initial condition is replaced by $y(0) = y_0$. Proceeding as in the previous example, (5.18) gives $c = -\frac{1}{y_0}$, and hence

$$y = \frac{y_0}{1 - y_0 t} \tag{5.22}$$

is the solution of the initial value problem (5.20) with the initial condition (5.21). Observe that the solution (5.22) becomes unbounded as $t \to 1/y_0$, so the interval of existence of the solution is

$$\begin{cases} -\infty < t < \frac{1}{y_0} & \text{if} \quad y_0 > 0 \\ \frac{1}{y_0} < t < \infty & \text{if} \quad y_0 > 0 \end{cases}$$

**Remark** The above two examples illustrates another feature of initial value problems for nonlinear equations; namely, the singularities of the solution may depend in an essential way on the initial conditions as well as on the differential equation.

**General Solution do not provide all the solutions**

Another way in which linear and nonlinear equations differ concerns the concept of a general solution.

1. For a first order linear equation it is possible to obtain a solution containing one arbitrary constant (called general solution), from which all possible solutions follow by specifying values for this constant.

2. For nonlinear equations this may not be the case; even though a solution containing an arbitrary constant may be found, there may be other solutions that cannot be obtained by giving values to this constant.

***Example* 8** $y = \psi(t) = 0$ for all $t$ is a solution of the differential equation

$$y' = y^2$$

that cannot be obtained from the solution (Ref. Example 6)

$$y = -\frac{1}{t+c}$$

by assigning a value to $c$.

**Remark** We use the term "general solution" only when discussing linear equations.

The above example leads to the following definition.

**Definition** (*Singular solution*) In some cases, there may be further solutions of a given differential equation, which cannot be obtained by assigning a definite value to the arbitrary constant in the general solution. Such a solution is called a **singular solution** of the differential equation.

***Example* 9** $y = \psi(t) = 0$ for all $t$ is a singular solution of the IVP

$$y' = y^2, \ y(0) = 1$$

as it cannot be obtained from the solution

$$y = -\frac{1}{t+c}$$

by assigning a value to $c$.

**Exercises**

In each of Exercises 1-6, determine (without solving the problem) an interval in which the solution of the given initial value problem is certain to exist.

1. $(t-3)y' + (\ln t)y = 2t, y(1) = 2$

2. $t(t-5)y' + y = 0, y(2) = 1$

3. $y' + (\tan t)y = \sin t, y(\pi) = 0$

4. $(4-t^2)y' + 2ty = 3t^2, y(-3) = 1$

5. $(4-t^2)y' + 2ty = 3t^2, y(1) = -3$

6. $(\ln t)y' + y = \cot t, y(2) = 3$

   In each of Exercises 7-12, state where in the $ty$-plane the hypotheses of Theorem are satisfied.

7. $y' = \frac{t-y}{2t+5y}$

8. $y' = (1-t^2-y^2)^{\frac{1}{2}}$

9. $y' = (t^2-y^2)^{\frac{3}{2}}$

10. $y' = \frac{\ln|ty|}{1-t^2+y^2}$

11. $\frac{dy}{dt} = \frac{1+t^2}{3y-y^2}$

12. $\frac{dy}{dt} = \frac{(\cot t)y}{1+y}$

In each of Exercises 13-16, solve the given initial value problem and determine how the interval in which the solution exists depends on the initial value $y_0$.

13. $y' = -4\frac{t}{y}, y(0) = y_0$

14. $y' = 2ty^2, y(0) = y_0$

15. $y' + y^3 = 0, y(0) = y_0$

16. $y' = \frac{t^2}{y(1+t^3)}, y(0) = y_0$

17. Consider the initial value problem $y' = y^{\frac{1}{3}},\ y(0) = 0$.

    (a) Is there a solution that passes through the point $(1,\ 1)$? If so, find it.

    (b) Is there a solution that passes through the point $(2,\ 1)$? If so, find it.

    (c) Consider all possible solutions of the given initial value problem. Determine the set of values that these solutions have at $t = 2$.

***Answers***

1. $0 < t < 3$

2. $0 < t < 5$

3. $\frac{\pi}{2} < t < \frac{3\pi}{2}$

4. $-\infty < t < -2$

5. $-2 < t < 2$
6. $1 < t < \pi$
7. $2t + 5y > 0$ or $2t + 5y < 0$
8. $t^2 + y^2 < 1$
9. Everywhere
10. $1 - t^2 + y^2 > 0$ or $1 - t^2 + y^2 < 0$, $t \neq 0$, $y \neq 0$
11. $y \neq 0, y \neq 3$
12. $y \neq n\pi$ for $n = 0,\ \pm 1,\ \pm 2,\ \ldots\ y \neq -1$
13. $y = \pm\sqrt{y_0^2 - 4t^2}$ if $y_0 \neq 0; |t| < \frac{|y_0|}{2}$
14. $y = [(1/y_0) - t^2]^{-1}$ if $y_0 \neq 0$; $y = 0$ if $y_0 = 0$;

    interval is $|t| < \frac{1}{\sqrt{y_0}}$ if $y_0 > 0$; $-\infty < t < \infty$ if $y_0 \leq 0$;
15. $y = y_0/\sqrt{2ty_0^2 + 1}$ if $y_0 \neq 0$; $y = 0$ if $y_0 = 0$;

    interval is $-\frac{1}{2y_0^2} < t < \infty$ if $y_0 \neq 0$; $-\infty < t < \infty$ if $y_0 = 0$
16. $y = \pm\sqrt{\frac{2}{3}\ln(1 + t^3) + y_0^2}$; $\quad -[1 - \exp(-3y_0^2/2)]^{1/3} < t < \infty$

Chapter 6

# Exact Differential Equations

In this chapter we consider a class of equations known as exact equations for which there is a well-defined method of solution. We begin with solving a differential equation that is neither linear nor separable.

## 6.1 Exact Differential Equations

***Example* 1** Solve the differential equation

$$2x + y^2 + 2xyy' = 0 \tag{6.1}$$

*Solution*

The equation is neither linear nor separable. However, we observe that the function $\psi(x, y) = x^2 + xy^2$ of two variables $x$

and $y$ has the property that

$$2x + y^2 = \frac{\partial \psi}{\partial x}, \qquad 2xy = \frac{\partial \psi}{\partial y} \tag{6.2}$$

Therefore the differential equation (6.1) can be written as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0. \tag{6.3}$$

Assuming that $y$ is a function of $x$, and using the chain rule for functions of one variable, we have

$$\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x}\frac{dx}{dx} + \frac{\partial \psi}{\partial y}\frac{dy}{dx}.$$

Hence we can write Eq.(6.3) in the equivalent form

$$\frac{d\psi}{dx} = 0$$

i.e.,

$$\frac{d\psi}{dx} = \frac{d}{dx}(x^2 + xy^2) = 0 \tag{6.4}$$

Therefore

$$\psi(x, y) = x^2 + xy^2 = c, \tag{6.5}$$

where $c$ is an arbitrary constant, is an equation that defines solutions of Eq.(6.1) implicitly.

**Method of Solving Equations of the Form** $M(x, y) + N(x, y)y' = 0$

Let the differential equation

$$M(x, y) + N(x, y)y' = 0 \tag{6.6}$$

be given. Suppose that we can identify a function $\psi$ such that

$$\frac{\partial \psi}{\partial x}(x, y) = M(x, y), \quad \frac{\partial \psi}{\partial y}(x, y) = N(x, y), \tag{6.7}$$

and such that $\psi(x, y) = c$ defines $y = \phi(x)$ implicitly as a differentiable function of $x$. Then

$$M(x, y) + N(x, y)y' = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = \frac{d}{dx}\psi[x, \phi(x)]$$

and the differential equation (6.6) becomes

$$\frac{d}{dx}\psi[x, \phi(x)] = 0. \tag{6.8}$$

In this case Eq.(6.6) is said to be an **exact differential equation.** Solutions of Eq.(6.6), or the equivalent Eq.(6.8), are given implicitly by

$$\psi(x, y) = c, \tag{6.9}$$

where $c$ is an arbitrary constant.

In Example 1 it was relatively easy to see that the differential equation was exact and, in fact, easy to find its solution, by recognizing the required function $\psi$. For more complicated equations it may not be possible to do this so easily. A systematic way of determining whether a given differential equation is exact is provided by the following theorem.

**Theorem** Let the functions $M$, $N$, $M_y$, and $N_x$ (where subscripts denote partial derivatives) be continuous in the rectangular region $R: \ \alpha < x < \beta, \ \ \gamma < y < \delta$. Then

$$M(x, y) + N(x, y)y' = 0 \tag{6.10}$$

is an exact differential equation in the region $R$ if and only if

$$M_y(x, y) = N_x(x, y) \tag{6.11}$$

at each point of $R$. That is, there exists a function $\psi$ satisfying

$$\psi_x(x, y) = M(x, y), \ \ \psi_y(x, y) = N(x, y), \tag{6.12}$$

if and only if $M$ and $N$ satisfy Eq.(6.11).

**Proof** The proof of this theorem has two parts. First we show that if there is a function $\psi$ such that Eqs.(6.12) are true, then it follows that Eq.(6.11) is satisfied. Computing $M_y$ and $N_x$ from Eqs.(6.12), we obtain

$$M_y(x, y) = \psi_{xy}(x, y), \qquad N_x(x, y) = \psi_{yx}(x, y). \tag{6.13}$$

Since $M_y$ and $N_x$ are continuous, it follows that $\psi_{xy}$ and $\psi_{yx}$ are also continuous. This guarantees their equality, and Eq.(6.11) follows.

We now show that if $M$ and $N$ satisfy Eq.(6.11), then Eq.(6.10) is exact. The proof involves the construction of a function $\psi$ satisfying Eqs.(6.12)

$$\psi_x(x,y) = M(x,y), \qquad \psi_y(x,y) = N(x,y).$$

We begin by integrating the first of Eqs. (6.12) with respect to $x$, holding $y$ constant. We obtain

$$\psi(x,y) = Q(x,y) + h(y), \tag{6.14}$$

where $Q(x,y)$ is any differentiable function such that $\partial Q(x,y)/\partial x = M(x,y)$. For example, we might choose

$$Q(x,y) = \int_{x_0}^{x} M(s,y)ds \tag{6.15}$$

where $x_0$is some specified constant in$\alpha < x_0 < \beta$. The function $h$ in Eq.(6.14) is an arbitrary differentiable function of $y$, playing the role of the arbitrary constant. Now we must show that it is always possible to choose $h(y)$ so that the second of Eqs(6.12) is satisfied, that is, $\psi_y = N$. By differentiating Eq.(6.14) with respect to $y$ and setting the result equal to $N(x,y)$, we obtain

$$\psi_y(x,y) = \frac{\partial Q}{\partial y}(x,y) + h'(y) = N(x,y).$$

Then, solving for $h'(y)$, we have

$$h'(y) = N(x, y) - \frac{\partial Q}{\partial y}(x, y). \tag{6.16}$$

In order for us to determine $h(y)$ from Eq.(6.16), the right side of Eq.(6.16), despite its appearance, must be a function of $y$ only. To establish that this is true, we can differentiate the quantity in question with respect to $x$, obtaining

$$\frac{\partial N}{\partial x}(x, y) - \frac{\partial}{\partial x}\frac{\partial Q}{\partial y}(x, y). \tag{6.17}$$

By interchanging the order of differentiation in the second term of Eq.(6.17), we have

$$\frac{\partial N}{\partial x}(x, y) - \frac{\partial}{\partial y}\frac{\partial Q}{\partial x}(x, y),$$

or, since $\partial Q/\partial x = M$,

$$\frac{\partial N}{\partial x}(x, y) - \frac{\partial M}{\partial y}(x, y),$$

which is zero on account of Eq.(6.11). Hence, despite its apparent form, the right side of Eq.(6.16) does not, in fact, depend on $x$. Then we find $h(y)$ by integrating Eq.(6.16), and upon substituting this function in Eq.(6.14), we obtain the required function $\psi(x, y)$. This completes the proof.

***Example* 2** Solve the differential equation

$$(y \cos x + 2xe^y) + (\sin x + x^2 e^y - 1)y' = 0 \tag{6.18}$$

*Solution*

Here taking

$M = y \cos x + 2xe^y$ and $N = \sin x + x^2 e^y - 1,$

we obtain

$$M_y(x, y) = \cos x + 2xe^y = N_x(x, y).$$

Hence the given equation is exact. Thus there is a $\psi(x, y)$ such that

$$\psi_x(x, y) = y \cos x + 2xe^y \tag{6.19}$$

and

$$\psi_y(x, y) = \sin x + x^2 e^y - 1. \tag{6.20}$$

Integrating (6.19) with respect to $x$, we obtain

$$\psi(x, y) = y \sin x + x^2 e^y + h(y), \tag{6.21}$$

where $h(y)$ is a function of $y$ alone.

Setting $\psi_y = N$ , from (6.21) and (6.20) we obtain

$$\psi_y(x, y) = \sin x + x^2 e^y + h'(y) = \sin x + x^2 e^y - 1.$$

Thus $h'(y) = -1$ and which on integration yields,

$$h(y) = -y.$$

The constant of integration can be omitted since any solution of the differential equation $h'(y) = -1$ is satisfactory; we do not require the most general one. Substituting for $h(y)$ in Eq.(6.21) gives

$$\psi(x, y) = y \sin x + x^2 e^y - y.$$

Hence solutions of Eq.(6.18) are given implicitly by

$$y \sin x + x^2 e^y - y = c \tag{6.22}$$

***Example* 3** (A differential equation that is not exact) Examine that differential equation

$$(3xy + y^2) + (x^2 + xy)y' = 0 \tag{6.23}$$

cannot be solved using the method discussed above.

*Solution*

Here taking

$M = 3xy + y^2$ and $N = x^2 + xy$,

we obtain

$$M_y(x, y) = 3x + 2y, \quad N_x(x, y) = 2x + y;$$

Since $M_y \neq N_x$, the given equation is **not exact.** Hence the procedure described above cannot produce a solution. We postpone the work of solving of this equation to the section *Integrating Factors*, but only see that it cannot be solved by the procedure described above. For this, let us seek a function $\psi$ such that

$$\psi_x(x,y) = 3xy + y^2 \tag{6.24}$$

and

$$\psi_y(x,y) = x^2 + xy. \tag{6.25}$$

Integrating , (6.24) yields

$$\psi(x,y) = \frac{3}{2}x^2y + xy^2 + h(y), \tag{6.26}$$

where $h$ is an arbitrary function of $y$ alone. To try to satisfy the equation (6.25), we compute $\psi_y$ from Eq.(6.26) and set it equal to $N$, yielding

$$\frac{3}{2}x^2 + 2xy + h'(y) = x^2 + xy$$

or

$$h'(y) = -\frac{1}{2}x^2 - xy. \tag{6.27}$$

Since the right side of Eq.(6.27) depends on both $x$ and $y$, it is impossible to solve Eq.(6.27) for $h(y)$. Thus there is no $\psi(x,y)$ satisfying both of Eqs.(6.24) and (6.25).

**Working Method for Solving the Exact Equation**

The following method helps to solve exact equation in an easy way.

If the differential equation

$$M(x,y) + N(x,y)y' = 0$$

or, equivalently written in the form

$$M\,dx + N\,dy = 0$$

satisfies the (necessary) condition

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

then it is **exact**. Then its solution is given by

$$\int M dx + \int (\text{terms in } N \text{ not involving } x)\,dy = c. \qquad (6.28)$$

***Example* 4** Show that the equation

$$(1 + 4xy + 2y^2)dx + (1 + 4xy + 2x^2)dy = 0$$

is exact and solve it.

*Solution*

Here $M = 1 + 4xy + 2y^2$and $N = 1 + 4xy + 2x^2$.

Differentiating $M$ partially with respect to $y$, we get

$$\frac{\partial M}{\partial y} = 4x + 4y$$

and differentiating $N$ partially with respect to $x$, we get

$$\frac{\partial N}{\partial x} = 4x + 4y,$$

so that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Hence, using (6.28),

$$\int (1 + 4xy + 2y^2)dx + \int (1)dy = c$$

i.e.,

$$x + 2x^2y + 2xy^2 + y = c$$

is the general solution.

***Example* 5** Solve the equation $(1 + y)\frac{dy}{dx} = 1 - x$.

*Solution*

The given equation is

$$(1 + y)dy = (1 - x)dx.$$

That is,

$$(1 - x)dx - (1 + y)dy = 0.$$

Then, $M = 1 - x$ and $N = -(1 + y)$ and

$$\frac{\partial M}{\partial y} = 0 = \frac{\partial N}{\partial x}$$

and hence the equation is exact and its solution is given by

$$x - \frac{x^2}{2} - y - \frac{y^2}{2} = c.$$

***Example* 6** Solve the initial value problem

$$(y-1)dx + (x-3)dy = 0; \quad y(0) = \frac{2}{3}.$$

*Solution*

Here $M = y - 1$ and $N = x - 3$, and since

$$\frac{\partial M}{\partial y} = 1 = \frac{\partial N}{\partial x},$$

the given differential equation is exact.
Hence, using (6.28), we obtain the general solution as

$$xy - x - 3y = c.$$

*Next to find the particular solution*

Using the initial condition that $y = \frac{2}{3}$at $x = 0$, we get the particular value of $c$ as $c = -2$, so the unique solution to the initial value problem is

$$xy - x - 3y = -2$$

or

$$xy - x - 3y + 2 = 0.$$

**Exercises**

Examine that the following equations are exact and then solve.

1. $\left(x + \frac{2}{y}\right) dy + ydx = 0$

2. $(y - x^3)\, dx + (x + y^3)\, dy = 0$

3. $(y + y\cos\, xy)\, dx + (x + x\cos xy)\, dy = 0$

4. $(\sin\, x\, \sin\, y - xe^y)\, dy = (e^y + \cos\, x\, \cos y)dx$

5. $\frac{dy}{dx} = -\frac{ax+by}{bx+cy}$

6. $(e^x \sin y - 2y \sin x)dx + (e^x \cos y + 2\cos\, x)dy = 0$

7. $(ye^{xy} \cos 2x - 2e^{xy} \sin\, 2x + 2x)dx + (xe^{xy} \cos 2x - 3)dy = 0$

8. $(y/x + 6x)dx + (\ln\, x - 2)dy = 0; \quad x > 0$

9. Solve the given initial value problem and determine at least approximately where the solution is valid.

$$(2x - y)dx + (2y - x)dy = 0, \quad y(1) = 0$$

10. Find the value of $b$ for which the given equation is exact, and then solve it using that values of $b$.

$$(xy^2 + bx^2y)dx + (x + y)x^2dy = 0$$

11. Assume that the equation

$$M(x, y) + N(x, y)y' = 0,$$

meets the requirements of Theorem 1 in a rectangle $R$ and is therefore exact. Show that a possible function $\psi(x, y)$is

$$\psi(x, y) = \int_{x_0}^{x} M(s, y_0)ds + \int_{y_0}^{y} N(x, t)dt,$$

where $(x_0, y_0)$ is a point in $R$.

12. Show that any separable equation

$$M(x) + N(y)y' = 0$$

is also exact.

***Answers***

1. $xy + \ln y^2 = c.$
2. $4xy - x^4 + y^4 = c.$
3. $xy + \sin xy = c.$
4. $xe^y + \sin x \cos y = c.$
5. $ax^2 + 2bxy + cy^2 = k$
6. $y = 0$ is a solution. General solution is $e^x \sin y + 2y \cos x = c$;
7. $e^{xy} \cos 2x + x^2 - 3y = c$
8. $y \ln x + 3x^2 - 2y = c$
9. $y = [x + \sqrt{28 - 3x^2}]/2, \quad |x| < \sqrt{28/3}$
10. $b = 3, \quad x^2y^2 + 2x^3y = c$

## 6.2 Integrating factors

Sometimes a given differential equation

$$M(x, y)dx + N(x, y)dy = 0 \tag{6.29}$$

may not be exact, but can be made exact by multiplying it by a suitable non-zero function $\mu(x, y)$. This function is called an **integrating factor** of equation (6.29). [Recall that this is the procedure that we used in solving linear differential equations].

**Method of Finding the Integrating Factor**

The integrating factor $\mu(x, y)$ of the differential equation

$$M(x, y)dx + N(x, y)dy = 0$$

is determined using solving the differential equation

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu.$$

If $\frac{M_y - N_x}{N}$ is a function of $x$ only, then there is an integrating factor $\mu(x, y) = \mu(x)$ that depends only on $x$; further $\mu(x)$ can be found by solving the above equation, which is both linear and separable.

***Example* 7** Solve the differential equation

$$x\,dy - y\,dx = 0. \tag{6.30}$$

*Solution*

The given differential equation is

$$-y\,dx + x\,dy = 0.$$

Here

$$M = -y, \quad N = x.$$

The above differential equation is not exact, since

$M_y = \frac{\partial M}{\partial y} = -1$ and $N_x = \frac{\partial N}{\partial x} = 1.$

Now

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu = \frac{-1-1}{x}\mu.$$

This implies

$$\frac{d\mu}{\mu} = -2\frac{dx}{x}..$$

Integrating,

$$\ln\mu = -2\ln x = \ln x^{-2} = \ln(\tfrac{1}{x^2}).$$

Hence $\mu = \frac{1}{x^2}$ is an integrating factor. Now, multiplying the given differential equation with the integrating factor, we obtain

$$\mu(-ydx + xdy) = 0$$

i.e.,

$$-\frac{y}{x^2}dx + \frac{1}{x}dy = 0.$$

On integrating, the above exact equation gives

$$\int \frac{-y}{x^2}dx + \int 0 \cdot dy = c.$$

i.e.,

$$\frac{y}{x} = c,$$

which is the general solution of the given differential equation.

***Example* 8** Make the following equation exact and hence solve:

$$y\,dx + (x^2y - x)dy = 0.$$

*Solution*

The given equation is not exact, since

$$\frac{\partial M}{\partial y} = 1 \neq 2xy - 1 = -\frac{\partial N}{\partial x}.$$

Now, we have

$$\frac{M_y - N_x}{N} = \frac{1 - (2xy - 1)}{x^2y - x} = \frac{-2(xy - 1)}{x(xy - 1)} = -\frac{2}{x},$$

which is a function only of $x$. Hence,

$$\mu = e^{\int -(2/x)\,dx} = e^{-2\ln x} = x^{-2}$$

is an integrating factor for the given differential equation. Multiplying the given equation by $x^{-2}$,

$$x^{-2}y\,dx + (y - x^{-1})dy = 0$$

and hence solution is

$$-x^{-1}y + \frac{y^2}{2} = c.$$

**II.** If the expression

$$\frac{M_y - N_x}{-M} \tag{6.31}$$

is a function of $y$ alone, say $h(y)$, then

$$\mu = e^{\int h(y)\,dy} \tag{6.32}$$

is also a function only of $y$ which satisfies the equation

$$\frac{1}{\mu}\left(N\frac{\partial\mu}{\partial x} - M\frac{\partial\mu}{\partial y}\right) = \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}.$$

and is hence an integrating factor for

$$M(x,\ y)\ dx + N(x,\ y)\,dy = 0.$$

**III.** There is another useful technique for converting simple nonexact equations into exact ones. We illustrate this in the following example.

***Example* 9** Solve the equation

$$y\ dx + (x^2y - x)\,dy = 0.$$

*Solution*

A rearrangement of the above differential equation gives

$$x^2y\,dy - (x\,dy - y\ dx) = 0. \tag{6.33}$$

We recall the differential formula

$$d\left(\frac{y}{x}\right) = \frac{x\,dy - y\ dx}{x^2}. \tag{6.34}$$

We divide Eq.(6.33) by $x^2$, and obtain it in the form

$$y\,dy - \frac{x\,dy - y\;dx}{x^2} = 0.$$

and using (6.34) this can be written as

$$ydy - d(y/x) = 0.$$

Hence general solution is

$$\frac{1}{2}y^2 - \frac{y}{x} = c.$$

**Some Differential Formulas**

In the above example, we have found an integrating factor for the given differential equation by noticing in it the combination $x\,dy - y\,dx$ and using $d(y/x) = (x\,dy - y\;dx)/x^2$ to exploit this observation. The following are some other differential formulas that are often useful in similar circumstances:

$$d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2} \tag{6.35}$$

$$d\,(x, y) = x\,dy + y\,dx \tag{6.36}$$

$$d\left(x^2 + y^2\right) = 2(x\,dx + y\,dx) \tag{6.37}$$

$$d\left(\tan^{-1}\frac{x}{y}\right) = \frac{ydx - xdy}{x^2 + y^2} \tag{6.38}$$

$$d\left(\ln\frac{x}{y}\right) = \frac{ydx - xdy}{xy} \tag{6.39}$$

We see from these formulas that the very simple differential equation

$$ydx - xdy = 0$$

has

$1/x^2,\ 1/y^2,\ 1/(x^2 + y^2)$, and $1/xy$

as integrating factors, and thus can be solved in this manner in a variety of ways.

**Exercises**

In Exercises 1-6, show that the given function is not exact but becomes exact when multiplied by the given integrating factor. Then solve the differential equation:

1. $2ydx + xdy = 0, \qquad x$

2. $\sin ydx + \cos ydy = 0, \qquad e^x$

3. $xdy - ydx = 0, \qquad \frac{1}{x^2}$

4. $y^2dx + (1 + xy)dy, \qquad e^{xy}$

5. $2\cos\pi ydx = \pi\sin\pi ydy, \qquad e^{2x}$

6. $y\cos xdx + 3\sin xdy = 0, \qquad y^2$

7. $x^2y^3 + x(1+y^2)y' = 0, \mu(x,y) = 1/xy^3$

8. $\left(\frac{\sin y}{y} - 2e^{-x}\sin x\right)dx + \left(\frac{\cos y + 2e^{-x}\cos x}{y}\right)dy = 0, \mu(x,y) = ye^x$

Find integrating factors and solve the following initial value problems.

9. $2dx + \sec x \cos y dy = 0, \qquad y(0) = 0$

10. $2x^2dx - 3xy^2dy = 0, \qquad y(1) = 0$

11. $2\sin y\ dx + \cos y\ dy = 0, \qquad \mathrm{y}(0) = \frac{\pi}{2}$

12. $(2y + xy) + 2xdy = 0, \qquad y(3) = \sqrt{2}$

13. Show that if $(N_x - M_y)/M = Q$, where Q is a function of $y$ only, then the differential equation

$$M + Ny' = 0$$

has an integrating factor of the form

$$\mu(y) = \exp \int Q(y)dy.$$

**Answers**

1. $x^2y = c$
2. $e^x \sin y = c$
3. $y = cx4.ye^{xy} = c$
4. $e^{2x} \cos \pi y = c$
5. $e^x \sin y = c$

6. $y = 0$ is a solution. General solution is $x^2 + 2\ln|y| - y^{-2} = c$.

7. $e^x \sin y + 2y \cos x = c$.

8. $\mu = \frac{1}{(y+1)^2}$ ; $x + 1 = c(y + 1)$

9. $\mu = \cos x$ ; $2\sin x + \sin y = 0$

10. $\mu = e^{2x}$; $e^{2x} \sin y = 1$

**Table**

| Differential equation | Integrating Factor | Exact Differential Equation | General solution |
|---|---|---|---|
| $x\,dy - y\,dx = 0$ | $\frac{1}{x^2}$ | $\frac{xdy - ydx}{x^2} = 0$ | $\frac{y}{x} = c$ |
| $x\,dy - y\,dx = 0$ | $\frac{1}{y^2}$ | $\frac{ydx - xdy}{y^2} = 0$ | $\frac{x}{y} = c$ |
| $x\,dy - y\,dx = 0$ | $\frac{1}{xy}$ | $\frac{xdy - ydx}{xy} = 0$ | $\ln \frac{y}{x} = c$ |
| $x\,dy - y\,dx = 0$ | $\frac{1}{x^2+y^2}$ | $\frac{xdy - ydx}{x^2+y^2} = 0$ | $\tan^{-1} \frac{y}{x} = c$ |
| $x\,dy - y\,dx = 0$ | $\frac{1}{(xy)^n}$ | $\frac{xdy + ydx}{xy} = 0*$ | $\ln xy = c*$ |
| $x\,dy - y\,dx = 0$ | $\frac{1}{(x^2+y^2)^n}$ | $\frac{xdx + ydy}{x^2+y^2} = 0*$ | $\ln\sqrt{x^2 + y^2} = c*$ |

*$for\ n=1$

Chapter 7

# Existence and Uniqueness of Solutions - Picard's Iteration Method

We begin with some initial value problems.

## 7.1 Existence and Uniqueness of Solutions

$(i)$ The initial value problem

$$|y'| + |y| = 0, \tag{7.1}$$

with the initial condition

$$y(0) = 1 \tag{7.2}$$

has no solution because $y \equiv 0$ is the only solution of the differential equation (7.1).

(*ii*) The initial value problem $y' = t$, $y(0) = 1$ has precisely one solution , namely $y = \frac{t^2}{2} + 1$.

(*iii*) The initial value problem $ty' = y - 1$, $y(0) = 1$ has infinitely many solutions, namely , $y = 1 + ct$, where $c$ is arbitrary.

**Remark** From these examples we see that an initial value problem

$$y' = f(t, y), \;\; y(t_0) = y_0 \tag{7.3}$$

may have none, one, or more than one solution. This leads to the following two fundamental questions.

**1. Problem of existence** Under what conditions does an initial value problem of the form (7.3) have at least one solution?

**2. Problem of uniqueness** Under what conditions does the problem have a unique solution, that is, only one solution?

**Existence and Uniqueness Theorem** If $f$ and $\frac{\partial f}{\partial y}$ are continuous in a rectangle $R: \quad |t| < a, \quad |y| < b$, then there is some interval $|t| \leq h < a$ in which there exists a unique solution $y = \phi(t)$ of the initial value problem

$$y' = f(t, y), \;\; y(0) = 0. \tag{7.4}$$

The solution mentioned in the Existence and Uniqueness Theorem has the form

$$\phi(t) = \int_0^t f(s,\ \phi(s))ds, \tag{7.5}$$

where we have made use of the initial condition $\phi(0) = 0$.

Since Eq. (7.5) contains an integral of the unknown function $\phi$, it is called an **integral equation.** This integral equation is not a formula for the solution of the initial value problem, but it does provide another relation satisfied by any solution of Eqs. (7.4). Conversely, suppose that there is a continuous function $y = \phi(t)$ that satisfies the integral equation (7.5); then, it can be shown that, this function also satisfies the initial value problem (7.4).

One method of showing that the integral equation (7.5) has a unique solution is known as the **method of successive approximations** or **Picard's iteration method.**

In Picard's iteration method, we start by choosing an initial function $\phi_0$, either arbitrarily or to approximate in some way the solution of the initial value problem. The simplest choice is

$$\phi_0(t) = 0; \tag{7.6}$$

then $\phi_0$ at least satisfies the initial conditions in Eqs.(7.4), although not the differential equation. The next approximation $\phi_1$ is obtained by substituting $\phi_0(s)$ for $\phi(s)$ in the right side of Eq.

(7.5) and calling the result of this operation $\phi_1(t)$. Thus

$$\phi_1(t) = \int_0^t f(s,\ \phi_0(s))ds.$$

Similarly, $\phi_2$ is obtained from $\phi_1$ using the following:

$$\phi_2(t) = \int_0^t f(s,\ \phi_1(s))ds.$$

In general,

$$\phi_{n+1}(t) = \int_0^t f(s,\ \phi_n(s))ds. \tag{7.7}$$

***Example* 1** Solve the initial value problem

$$y' = 2t(1+y), \ \ y(0) = 0 \tag{7.8}$$

by the method of successive approximations.

*Solution*

Note first that if $y = \phi(t)$, then the corresponding integral equation is

$$\phi(t) = \int_0^t 2s[1+\phi(s)]ds. \tag{7.9}$$

If the initial approximation is $\phi_0(t) = 0$, it follows that

$$\phi_1(t) = \int_0^t 2s[1+\phi_0(s)]ds = \int_0^t 2sds = t^2. \tag{7.10}$$

Similarly,

$$\phi_2(t) = \int_0^t 2s[1+\phi_1(s)]ds = \int_0^t 2s[1+s^2]ds = t^2 + \frac{t^4}{2} \quad (7.11)$$

and

$$\phi_3(t) = \int_0^t 2s[1+\phi_2(s)]ds = \int_0^t 2s\left[1+s^2+\frac{s^4}{2}\right]ds = t^2+\frac{t^4}{2}+\frac{t^6}{2\cdot 3}. \quad (7.12)$$

Equations (7.10), (7.11), and (7.12) suggest that

$$\phi_n(t) = t^2 + \frac{t^4}{2!} + \frac{t^6}{3!} + \cdots + \frac{t^{2n}}{n!} \quad (7.13)$$

for each $n \geq 1$, and this result can be established by mathematical induction, as follows. Equation (7.13) is certainly true for $n = 1$; see Eq. (7.10). We must show that if it is true for $n = k$, then it also holds for $n = k + 1$. We have

$$\begin{aligned} \phi_{k+1}(t) &= \int_0^t 2s[1+\phi_k(s)]ds \\ &= \int_0^t 2s\left(1+s^2+\frac{s^4}{2!}+\cdots+\frac{s^{2k}}{k!}\right)ds \\ &= t^2+\frac{t^4}{2!}+\frac{t^6}{3!}+\cdots+\frac{t^{2k+2}}{(k+1)!} \end{aligned} \quad (7.14)$$

and the inductive proof is complete.

**Remark to Example** It follows from Eq. (7.13) that $\phi_n(t)$ is the $n$th partial sum of the infinite series

$$\sum_{k=1}^{\infty} \frac{t^{2k}}{k!}; \tag{7.15}$$

hence $\lim_{n\to\infty} \phi_n(t)$ exists if and only if the series (7.15) converges. Applying the ratio test, we see that, for each $t$,

$$\left| \frac{t^{2k+2}\ k!}{(k+1)!t^{2k}} \right| = \frac{t^2}{k+1} \to 0 \text{ as } k \to \infty. \tag{7.16}$$

Thus the series (7.15) converges for all $t$, and its sum $\phi(t)$ is the limit of the sequence $\{\phi_n(t)\}$. Further, since the series (7.15) is a Taylor series, it can be differentiated or integrated term by term as long as $t$ remains within the interval of convergence, which in this case is the entire $t$-axis. Therefore, we can verify by direct computation that $\phi(t) = \sum_{k=1}^{\infty} \frac{t^{2k}}{k!}$ is a solution of the integral equation (7.9). Alternatively by substituting $\phi(t)$for $y$ in Eqs. (7.8), we can verify that this function satisfies the initial value problem. Identifying $\phi$ in terms of elementary functions, we have the solution of the given IVP as $\phi(t) = e^{t^2} - 1$.

**Working Rule (Picard's Iteration Method)**

Consider the initial value problem

$$y' = f(t,\ y), y(t_0) = y_0. \tag{7.17}$$

Then Picard's iterative formula is

$$y_{n+1} = y_0 + \int_{t_0}^{t} f(t,\ y_n)dt(n = 1,\ 2,\ 3,\ \dots\ ) \qquad (7.18)$$

***Example* 2** Find approximate solutions by Picard's iteration method to the initial value problem $y' = 1 + y^2$ with the initial condition $y(0) = 0$. Hence find the approximate value of $y$ at $t = 0.1$ and $t = 0.2$.

**Solution** Picard's iteration's $(n+1)^{th}$ step is given by (7.18).

In this problem

$$f(t,\ y) = 1 + y^2; t_0 = 0, \quad y_0 = y(t_0) = y(0) = 0,$$

and hence

$$f(t,\ y_n) = 1 + y_n^2.$$

Substituting these values in (7.18), we obtain

$$y_{n+1} = 0 + \int_0^t \left(1 + y_n^2\right) dt(n = 1,\ 2,\ 3,\ \dots\ )$$

i.e., $y_{n+1} = t + \int_0^t y_n^2 dt\ (n = 1,\ 2,\ 3,\ \dots\ )$

**Step 1** $(n = 0)$:

$$y_1 = t + \int_0^t y_0^2 dt$$

Putting $y_0 = 0$, we obtain

$$y_1 = t + \int_0^t 0^2 dt = t.$$

**Step 2** $(n = 1)$:

$$y_2 = t + \int_0^t y_1^2 dt$$

Putting $y_1 = t$, we obtain

$$y_2 = t + \int_0^t t^2 dt = t + \frac{1}{3}t^3.$$

**Step 3** $(n = 2)$:

$$y_3 = t + \int_0^t y_2^2 dt$$

Putting $y_2 = t + \frac{1}{3}t^3$, we obtain

$$y_3 = t + \int_0^t \left(t + \frac{1}{3}t^3\right)^2 dt$$

$$= t + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \frac{1}{63}t^7.$$

We can continue the process. But we take the above as an approximate solution to the given initial value problem. That is,

$$y = y(t) = t + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \frac{1}{63}t^7. \tag{7.19}$$

Substituting $t = 0.1$, and $t = 0.2$, in (7.19), we obtain

$$y(0.1) = 0.100334$$

and

$$y(0.2) = 0.202709.$$

The above are not exact values for $y$ at the given $x$ points, but the approximate values.

**Remark to Example** The exact solution to the initial value problem $y' = 1 + y^2$; $y(0) = 0$ can be obtained by separating variables and the solution is given by

$$y(t) = \tan t.$$

Also the series corresponding to $\tan x$ is

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \dots \qquad \left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right)$$

i.e.,

$$y(x) = \tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \dots \tag{7.20}$$

The first three terms of $y_3$ and the series in (7.20) are the same. The series in (7.20) converges for $|x| < \frac{\pi}{2}$, and we can expect that our sequence $y_1,\ y_2,\ \dots$ converges to a function which is the solution of our problem for $|x| < \frac{\pi}{2}$.

**Exercises**

In Exercises 1-7, solve the initial value problem by Piacrd's iteration method (Do three steps). Also find the exact solution.

1. $y' = y, \ \ y(0) = 1.$
2. $y' = x + y, \ \ y(0) = -1.$
3. $y' = xy + 2x - x^3, \ \ y(0) = 0.$
4. $y' = y - y^2, \ \ y(0) = \frac{1}{2}.$
5. $y' = y^2, \ \ y(0) = 1.$
6. $y' = 2\sqrt{y}, \ \ y(1) = 0.$
7. $y' = \frac{3y}{x}, \ \ y(1) = 1.$

   In Exercises 8-15, solve the initial value problem by Picard's iteration method (Do four steps). Also find the value of $y$ at the given points of $x$.

8. $y' = 2x - y, \ \ y(1) = 3.$ Also find $y(1.1)$.
9. $y' = x - y, \ \ y(0) = 1.$ Also find $y(0.2)$.
10. $y' = x^2y, \ \ y(1) = 2.$ Also find $y(1.2)$.
11. $y' = 3x + y^2, \ \ y(0) = 1.$ Also find $y(0.1)$.
12. $y' = 2x + 3y, \ \ y(0) = 1.$ Also find $y(0.25)$.
13. $2\frac{dy}{dx} = x + y, \ \ y(0) = 2.$ Also find $y(0.1)$.
14. $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}, \ \ y(1) = 1.$ Also find $y(1.1)$.

15. $\frac{dy}{dx} - xy = 1, \ y(0) = 1$. Also find $y(1)$. Compare with the exact solution and exact value.

16. Obtain the approximate solution of

$$\frac{dy}{dx} = x + x^4 y, \ \ y(0) = 3$$

by Picard's iteration method. Tabulate the values of $y$, for $x = 0.1(0.1)0.5, \ 3D$.

**Attention!** In Exercise 13-15, first bring the differential equation to the standard form (7.17) before starting Picard's iteration procedure.

***Answers***

1. The approximations approach the exact solution $y = e^x$.
2. $y_0 = 1, \ y_{-n} = -1 - x + \frac{x^{n+1}}{(n+1)!}, \ \ y = -1 - x$
3. $y_n = x^2 - \frac{x^{2n+2}}{2^n (n+1)!}, \ \ y = x^2$
4. $y_0 = \frac{1}{2}, \ y_1 = \frac{1}{2} + \frac{x}{4}, \ y_2 = \frac{1}{2} + \frac{x}{4} - \frac{x^3}{48}, \ \ y_3 = \frac{1}{2} + \frac{x}{4} - \frac{x^3}{48} + \frac{x^5}{480} - \frac{x^7}{16128}$ Exact solution is $y = \frac{1}{1+e^{-x}}$

8 $y_n = 2 + x^2 - \int_1^x x y_{n-1} dx, y_1 = x^2 - 3x + 5,$

$y_2 = -\frac{x^3}{3} + \frac{5}{2}x^2 - 5x + \frac{35}{6}; \ y(1) = 2.9147$

9 $y_4 = 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{12} - \frac{x^5}{120}; \ \ y(0.2) = 0.837.$

10 2.544

11 1.1264

12 2.1826

13 $y_2 = 3 + \frac{x^2}{2} + \frac{3x^5}{5} + \frac{x^7}{14} + 3\frac{x^{10}}{50}$; 2.105

14 0.995

15 $y_n(x) = 1 + x + \int_0^x x y_{n-1}(x) dx, \quad y_1(x) = 1 + x + \frac{x^2}{2}$ *etc.*
Exact solution (considering the given equation as linear differential equation) is $y = e^{\frac{x^2}{2}} \left( \int_0^x e^{\frac{x^2}{2}} dx + 1 \right)$

Chapter 8

# Differential Equations of Second Order

## 8.1 Differential Equations of Second Order

In this chapter we consider the solution of second order differential equations.

**Definition** A second order differential equation is said to be linear if it can be written in the form

$$y'' + p(t)y' + q(t)y = r(t). \tag{8.1}$$

If $r(t) = 0$ for every $t$, then equation (8.1) is said to be **homoge-**

**neous**; otherwise called **nonhomogeneous**. For example,

$$y'' + 4ty' + (t^2 - 3)y = 0$$

is a homogeneous equation and

$$y'' + 4y' + t^2 y = e^{-t} \sin t$$

is a nonhomogeneous equation.

Any differential equation of the second order which cannot be written in the form (8.1) is said to be **nonlinear**. For example,

$$y''y + y' = 0$$

and

$$y'' + (ty')^2 = \cos t$$

are nonlinear differential equations.

The functions $p$ and $q$ in (8.1) are called coefficients of the equation.

**Remark** Instead of Eq. (8.1) we may have equations of the form

$$P(t)y'' + Q(t)y' + R(t)y = G(t). \tag{8.2}$$

If $P(t) \neq 0$, we can divide Eq.(8.2) by $P(t)$ and obtain Eq. (8.1) with

$$p(t) = \frac{Q(t)}{P(t)}, q(t) = \frac{R(t)}{P(t)}, r(t) = \frac{G(t)}{P(t)}.$$

**Definition** (*Solution of a second order differential equation*) A function

$$y = \varphi(t)$$

is called a **solution** of a linear or nonlinear differential equation of the second order on some interval, say $a < t < b$ (perhaps infinite), if $\varphi(t)$ is defined and twice differentiable throughout that interval and is such that the equation becomes an identity when we replace the unspecified functions $y$ and its derivatives by $\varphi$ and its corresponding derivatives.

***Example* 1** Verify that the functions $y = \cos t$ and $y = \sin t$ are solutions of the homogeneous linear differential equation $y'' + y = 0$.

*Solution*

$y = \cos t$ is defined and twice differentiable for all t. Also $y' = -\sin t$ and $y'' = \cos t$, and substitution of these values in the given differential equation gives

$$-\cos t + \cos t = 0,$$

an identity. Hence $y = \cos t$ is a solution to the given differential equation. Similarly, it can be shown that $y = \sin t$ is also a solution of the given differential equation.

**Theorem Fundamental Theorem (Principle of Superposition)**

If $y_1$ and $y_2$ are two solutions of the linear homogeneous differential equation

$$L[y] = y'' + p(t)y' + q(t)y = 0, \tag{8.3}$$

then the linear combination $c_1y_1 + c_2y_2$ is also a solution for any values of the constants $c_1$ and $c_2$.

***Example* 2** Verify that the functions $y = 7\cos t$ and $y = \sqrt{2}\cos t + 9\sin t$ are solutions of the homogeneous linear differential equation $y'' + y = 0$.

*Solution*

By Example 1,$y = \cos t$ and $y = \sin t$ are solutions of the given differential equation. Hence, by Theorem any linear combination of $\cos t$ and $\sin t$, in particular $y = 7\cos t$ and$y = \sqrt{2}\cos t + 9\sin t$, is a solution of the differential equation.

**Attention:** The fundamental theorem does **not hold** for *nonhomogeneous linear differential equations* or for *nonlinear differential equations.* We illustrate this in the following examples.

***Example* 3** (*A nonhomogeneous linear differential equation for which fundamental theorem fails)*

Substitution shows that the functions

$$y = 1 + \cos t$$

and

$$y = 1 + \sin t$$

are solutions of the nonhomogeneous linear differential equation

$$y'' + y = 1,$$

**but** the functions

$2_{-}(1 + \cos t)$ and $(1 + \cos t) + (1 + \sin t)$

are **not** solutions of this differential equation, showing that fundamental theorem *fails* in the case of nonhomogeneous linear differential equations.

***Example* 4** (*A nonlinear differential equation for which fundamental theorem fails*)

Substitution shows that the functions

$y = t^2$ and $y = 1$

are solutions of the nonlinear differential equation

$$y''y - ty' = 0,$$

**but** the functions

$-t^2$ and $t^2 + 1$

are **not** solutions of this differential equation, showing that fundamental theorem *fails* in the case of nonlinear differential equation.

**General Solution, Basis**

**Definition** Two functions $y_1(t)$ and $y_2(t)$ are said to be **linearly dependent** on an interval $I$ where both functions are defined, if they are proportional on $I$, i.e., if

$$(a)\ y_1(t) = ky_2(t) \ \text{ or } \ (b)\ y_2(t) = ly_1(t)$$

holds for all $t$ on $I$; here $k$ and $l$ are numbers, zero or not. If the functions are not proportional on $I$, they are said to be **linearly independent**.

**Note** If one of the functions $y_1$ and $y_2$ is identically zero on $I$, then the functions are linearly dependent on $I$.

***Example* 5** The functions

$y_1 = 6t$ and $y_2 = 4t$

are linearly dependent on any interval, since $y_1 = \frac{6}{4}y_2$.

But the functions

$y_1 = t^2$ and $y_2 = t$

are linearly independent on any interval because $\frac{y_1}{y_2} = t \neq k$,(i.e., $\frac{y_1}{y_2}$ is not constant), so that no relation of the form $y_1 = ky_2$ or $y_2 = ly_1$ can hold.

Similarly, it can be seen that

(a) $\cos t$ and $\sin t$ are linearly independent

(b) $e^{2t}$ and $e^{-2t}$ are linearly independent

(c) The functions $y_1 = t$ and $y_2 = t + 1$are linearly independent on any interval.

**Definition** (*Basis*) Two linearly independent solutions of a linear homogeneous second order differential equation on an interval $I$ is called a **basis** or a **fundamental system** of solutions on $I$.

**Theorem 2** A solution

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

($c_1$, $c_2$ arbitrary constants) is a **general solution** of the homogeneous second order differential equation

$$y'' + p(t)y' + q(t)y = 0$$

on an interval $I$ of the $t$-axis if and only if the functions $y_1$ and $y_2$ constitute a **basis of solutions** of the differential equation on $I$, i.e., if and only if they are not proportional on $I$.

***Example* 6** The functions $y_1 = e^{-t}$ and $y_2 = e^{2t}$ are solutions* of the equation $y'' - y' - 2y = 0$. Since $\frac{y_1}{y_2} = e^{-3t}$ is not a constant, the solutions are linearly independent; hence they constitute a basis, and the corresponding general solution for all $t$ is

$$y = c_1 y_1 + c_2 y_2 = c_1 e^{-t} + c_2 e^{2t}.$$

*The method of finding the solutions will be discussed later.

***Example* 7** The functions $y_1 = e^t$ and $y_2 = 3e^t$ are solutions of the equation $y'' - 2y' + y = 0$. Since $y_2 = 3y_1$,the solutions are linearly dependent; hence they do not form a basis.

**Initial Value Problems and Boundary Value Problems**

Since the general solution to a second order differential equation

$$y'' + p(t)y' + q(t)y = 0. \tag{8.4}$$

contains *two* arbitrary constants, we need *two* conditions for obtaining a particular solution. In some cases these two conditions are of the type

$$y(t_0) = K, \qquad y'(t_0) = L \tag{8.5}$$

where $t = t_0$ is a given point and $K$ and $L$ are given numbers. The conditions in (8.5) are called **initial conditions**. Equation (8.4) and conditions (8.5) together constitute what is known as an **initial value problem**.

Some times the *two* conditions are of the type

$$y(A) = k_1, \qquad y(B) = k_2. \tag{8.6}$$

These are called **boundary conditions** since they refer to the end points (boundary points)$A$, $B$ of an interval $I$. Eq. (8.4) and conditions (8.6) together constitute what is known as a **boundary value problem**.

***Example* 8** (*Initial Value Problem)* Solve the initial value problem

$$y'' - y' - 2y = 0, \quad y(0) = 4, \quad y'(0) = 1\,,$$

where it is given that $y_1 = e^{-x}$ and $y_2 = e^{2x}$ form a fundamental system of solutions of the given differential equation.

*Solution*

A general solution[1] of the equation $y'' - y' - 2y = 0$ is given by

$$y = y(t) = c_1 e^{-t} + c_2 e^{2t}.$$

Hence, by the first initial condition,

$$y(0) = c_1 e^0 + c_2 e^0,$$

and so

$$c_1 + c_2 = 4. \tag{8.7}$$

Differentiation of the general solution gives the function

$$y' = y'(t) = -c_1 e^{-t} + 2c_2 e^{2t}.$$

$$y'(0) = -c_1 e^0 + 2c_2 e^0$$

and so

$$1 = -c_1 + 2c_2. \tag{8.8}$$

Solving the Eqs. (8.7) and (8.8), we obtain

$$c_1 = \frac{7}{3}, \quad c_2 = \frac{5}{3}.$$

[1]Method of finding general solution of a second order differential equation will be discussed in the coming chapters.

Hence the solution to the initial value problem is given by

$$y = y(t) = \frac{7}{3}e^{-t} + \frac{5}{3}e^{2t}.$$

# Chapter 9

# Solutions of Linear Homogeneous Equations-Wronskian

In this chapter we give a general theory for the solution of homogeneous equations

$$y'' + p(t)y' + q(t)y = 0 \tag{9.1}$$

with continuous, but otherwise arbitrary variable coefficients $f$ and $g$.

## 9.1 Linear Homogeneous Equations

**Theorem 1** (*Existence and Uniqueness Theorem for Initial Value Problems*) consider the initial value problem

$$y'' + p(t)y' + q(t)\,y = g(t), \qquad y(t_0) = y_0,\;\; y'(t_0) = y_0' \qquad (9.2)$$

where $p, q$ and $g$ are continuous functions on some open interval $I$ that contains the point $t_0$ . Then there exactly one solution $y = \phi(t)$,of this problem and the solution exists throughout the interval $I$.

***Example* 1** Find the longest interval in which the solution of the initial value problem

$$(t^2 - 3t)y'' + ty' - (t+3)y = 0, \quad y(1) = 2, \quad y'(1) = 1$$

is certain to exist.

*Solution*

If the given differential equation is written in the form of Eq.(9.2), then

$$p(t) = \frac{1}{t-3}, \;\; q(t) = -\frac{t+3}{t(t-3)}, \;\; \text{and} \;\; g(t) = 0.$$

The only points of discontinuity of the coefficients are $t = 0$ and $t = 3$. Therefore, the longest open interval, containing the initial point $t = 1$, in which all the coefficients are continuous is $0 < t < 3$. Thus, this is the longest interval in which Theorem guarantees that the solution exists.

***Example* 2** Find the unique solution of the initial value problem

$$y'' + p(t)y' + q(t)y = 0, y(t_0) = 0, y'(t_0) = 0,$$

where $p$ and $q$ are continuous in an open interval $I$ containing $t_0$.

*Solution*

Obviously the function $y = \phi(t) = 0$ for all $t$ in $I$ satisfies the differential equation and initial conditions. By the uniqueness part of Theorem, it is the only solution of the given problem.

**Theorem 2 Fundamental Theorem (Principle of Superposition)** If $y_1$ and $y_2$ are two solutions of the linear homogeneous differential equation

$$L[y] = y'' + p(t)y' + q(t)y = 0, \tag{9.3}$$

then the linear combination $c_1y_1 + c_2y_2$ is also a solution for any values of the constants $c_1$ and $c_2$.

**Proof**

$$\begin{aligned}
L[c_1y_1 + c_2y_2] &= [c_1y_1 + c_2y_2]'' + p[c_1y_1 + c_2y_2]' + q[c_1y_1 + c_2y_2] \\
&= c_1y_1{}'' + c_2y_2{}'' + c_1py_1{}' + c_2py_2{}' + c_1qy_1 + c_1qy_1 + c_2qy_2 \\
&= c_1[y_1{}'' + py_1{}' + qy_1] + c_2[y_2{}'' + py_2{}' + qy_2] \\
&= c_1L[y_1] + c_2L[y2] \\
&= c_1 \cdot 0 + c_2 \cdot 0 \ \text{ since } L[y_{\text{-}1}] = 0 \text{ and } L[y_{\text{-}2}] \\
&= 0.
\end{aligned}$$

Hence the theorem.

**Remark** A special case of Theorem occurs if either $c_1$ or $c_2$ is zero. Then we conclude that any constant multiple of a solution of Eq.(9.3) is also a solution.

**Definition** (*Wronskian*) **Wronskian** of two solutions $y_1$ and $y_2$ of the homogeneous differential equation

$$y'' + p(t)y' + q(t)y = 0 \tag{9.4}$$

is defined by

$$W[y_1,\ y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'.$$

**Theorem 3** Suppose that $y_1$ and $y_2$ are two solutions of

$$L[y] = y'' + p(t)y' + q(t)y = 0 \tag{9.5}$$

and that the initial conditions

$$y(t_0) = y_0, \quad y'(t_0) = y_0' \tag{9.6}$$

are assigned. Then it is always possible to choose the constant $c_1,\ c_2$ so that

$$y = c_1y_1(t) + c_2y_2(t)$$

satisfies the differential equation (9.5) and the initial conditions

(9.6) if and only if the Wronskian

$$W[y_1,\ y_2] = y_1y_2' - y_1'y_2$$

is not zero at $t_0$.

***Example* 3** $y_1(t) = e^{-2t}$ and $y_2(t) = e^{-3t}$are solutions of the differential equation

$$y'' + 5y' + 6y = 0.$$

Find the Wronskian of $y_1$ and $y_2$. Comment.

*Solution*

The Wronskian of these two functions is

$$W = \begin{vmatrix} e^{-2t} & e^{-3t} \\ -2e^{-2t} & -3e^{-3t} \end{vmatrix} = -e^{-5t}.$$

Since $W$ is nonzero for all values of $t$, the functions $y_1$ and $y_2$ can be used to construct solutions of the given differential equation, together with initial conditions prescribed at any value of $t$.

The next theorem justifies the term "general solution" that we are used for the linear combination$c_1y_1 + c_2y_2$.

**Theorem 4** Suppose that $y_1$ and $y_2$ are two solutions of the differential equation

$$L[y] = y'' + p(t)y' + q(t)y = 0. \tag{9.7}$$

Then the family of solutions

$$y = c_1 y_1(t) + c_2 y_2(t)$$

with arbitrary coefficients $c_1$ and $c_2$ includes every solution of Eq.(9.7) if and only if there is a point $t_0$ where the Wronskian of $y_1$ and $y_2$ is not zero.

Theorem states that, if and only if the wronskian of $y_1$ and $y_2$ is not everywhere zero, then the linear combination $c_1y_1 + c_2y_2$ contains all solutions of Eq.(9.7). It is therefore natural (and we have already done this in the preceding chapter) to call the expression

$$y = c_1 y_1(t) + c_2 y_2(t),$$

with arbitrary constant coefficients, **general solution** of Eq.(9.7). The solutions $y_1$ and $y_2$ are said to form a **fundamental set of solutions** of Eq.(9.7) if and only if their wronskian is nonzero.

***Example* 4** The functions $y_1 = t$ and $y_2 = t + 1$ from a fundamental set of solutions, since

$$W(y_1,\ y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t & 1+t \\ 1 & 1 \end{vmatrix} = t - (1+t) = 1 \neq 0.$$

***Example* 5** $y_1(t) = \cos\omega t$ and $y_2(t) = \sin\omega t$ are solutions of

$y'' + \omega^2 y = 0$. Their Wronskian is

$$\begin{aligned} W(y_1,\ y_2) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos\omega t & \sin\omega t \\ -\omega\sin\omega t & \omega\cos\omega t \end{vmatrix} \\ &= \omega(\cos^2\omega t + \sin^2\omega t) = \omega. \end{aligned}$$

Hence $y_1$ and $y_2$ are fundamental set of solutions (i.e., linearly independent) if and only if $\omega \neq 0$.

***Example* 6** The solutions $e^{-4t}$ and $te^{-4t}$ are fundamental set of solutions of $y'' + 8y' + 16y = 0$, since

$$W(y_1,\ y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-4t} & xe^{-4t} \\ -4e^{-4t} & e^{-4t} - 4te^{-4t} \end{vmatrix} \neq 0$$

for any value of $t$. Hence a general solution of $y'' + 8y' + 16y = 0$ on any interval $y = (c_1 + c_2 t)e^{-4t}$.

***Example* 7** Suppose that $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$ are two solutions of an equation of the form

$$y'' + f(t)y' + g(t)y = 0$$

Show that they form a fundamental set of solutions if $r_1 \neq r_2$.

*Solution*

The Wronskian of $y_1$ and $y_2$ is

$$\begin{aligned} W(y_1,\ y_2) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} \\ &= (r_2 - r_1)\exp[(r_1 + r_2)t]. \end{aligned}$$

Since the exponential function is never zero, and since we are assuming that $r_2 - r_1 \neq 0$, it follows that $W$ is nonzero for every value of $t$. Consequently, $y_1$ and $y_2$ form a fundamental set of solutions.

***Example* 8** Show that $y_1(t) = t^{1/2}$ and $y_2(t) = t^{-1}$ form fundamental set of solutions of

$$2t^2 y'' + 3ty' - y = 0, \qquad t > 0.$$

*Solution*

Since

$y_1{}'(t) = \frac{1}{2}t^{-1/2}$ and $y_1{}''(t) = -\frac{1}{4}t^{-3/2}$,

we have

$$2t^2\left(-\frac{1}{4}t^{-3/2}\right) + 3t\left(\frac{1}{2}t^{-1/2}\right) - t^{1/2} = \left(-\frac{1}{2} + \frac{3}{2} - 1\right)t^{1/2} = 0.$$

Hence $y_1(t) = t^{1/2}$ is a solution. Similarly, it can be seen that $y_2(t) = t^{-1}$ is a solution. It remains to show that they are linearly independent.

The Wronskian of $y_1$ and $y_2$ is

$$W(y_1,\ y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t^{1/2} & t^{-1} \\ \frac{1}{2}t^{-1/2} & -t^{-2} \end{vmatrix} = -\frac{3}{2}t^{-3/2}.$$

Since $W \neq 0$ for $t > 0$, it follows $y_1$ and $y_2$ form a fundamental set of solutions.

**Theorem 5** (*Existence of a general solution*)

Consider the differential equation

$$L[y] = y'' + p(t)y' + q(t)y = 0 \tag{9.8}$$

whose coefficients $p(t)$ and $q(t)$ are continuous on some open interval $I$. Choose some point $t_0$ in $I$. Let $y_1$ be solution of Eq. (9.8) that also satisfies the initial conditions

$$y(t_0) = 1, \qquad y'(t_0) = 0,$$

and let $y_2$ be the solution of Eq. (9.8) that satisfies the initial conditions

$$y(t_0) = 0, \qquad y'(t_0) = 1.$$

Then $y_1$ and $y_2$ form a fundamental set of solutions.

The following example illustrates that a given differential equation has more than one fundamental set of solutions; indeed, it has infinitely many.

***Example* 9** Find the fundamental set of solutions specified by the previous Theorem for the differential equation

$$y'' - y = 0, \tag{9.9}$$

using the initial point $t_0 = 0$.

*Solution*

The characteristic equation is $\lambda^2 - 1 = 0$. Hence $\lambda = \pm 1$. Thus two solutions of the given differential equation are

$y_1(t) = e^t$ and $y_2(t) = e^{-t}$.

Also, $W(y_1,\ y_2) = -2 \neq 0$, so $y_1$ and $y_2$ form a fundamental set of solutions. However they are not the fundamental set of solutions indicated by the previous Theorem because they do not satisfy the initial conditions mentioned in that theorem at the point $t_0 = 0$.

To find the fundamental solutions specified by the theorem, we need to find the solutions satisfying the proper initial conditions. Let us denote by $y_3(t)$ the solution of Eq.(9.9) that satisfies the initial conditions

$$y(0) = 1, \qquad y'(0) = 0. \tag{9.10}$$

The general solution of Eq.(9.9) is

$$y = c_1 e^t + c_2 e^{-t}, \tag{9.11}$$

and the initial conditions (9.10) are satisfied if $c_1 = 1/2$ and $c_2 = 1/2$. Thus

$$y_3(t) = \frac{1}{2}e^t + \frac{1}{2}e^{-t} = \cosh t.$$

Similarly, if $y_4(t)$ satisfies the initial conditions

$$y(0) = 0, \qquad y'(0) = 1, \tag{9.12}$$

then

$$y_4(t) = \frac{1}{2}e^t - \frac{1}{2}e^{-t} = \sinh t.$$

Since Wronskian of $y_3$ and $y_4$ is

$$W(y_1,\ y_2)(t) = \cosh^2 t - \sinh^2 t = 1 \neq 0,$$

these functions also form a fundamental set of solutions. Therefore, the general solution of Eq. (9.9) can be written as

$$y = k_1 \cosh t + k_2 \sinh t, \tag{9.13}$$

as well as in the form (9.11). We have $k_1$ and $k_2$ for the arbitrary constants Eq.(9.13) because they are not the same as the constants $c_1$ and $c_2$ in Eq. (9.11).

**Theorem 6 (Abel's Theorem)**

If $y_1$ and $y_2$ are the solutions of the differential equation

$$L[y] = y'' + p(t)y' + q(t)y = 0 \tag{9.14}$$

where $p$ and $q$ are continuous on an open interval $I$, then the

Wronskian $W(y_1, y_2)(t)$ is given by

$$W(y_1, y_2)(t) = c \exp\left[-\int p(t)dt\right], \tag{9.15}$$

where $c$ is a certain constant that depends that depends on $y_1$ and $y_2$, but not on $t$. Further, $W(y_1, y_2)(t)$ either is zero for all $t$ in $I$ (if $c = 0$) or else is never zero in $I$ (if $c \neq 0$).

**Proof** We start by noting that $y_1$ and $y_2$ satisfy

$$y_1'' + p(t)y_1' + q(t)y_1 = 0,$$

$$y_2'' + p(t)y_2' + q(t)y_2 = 0. \tag{9.16}$$

If we multiply the first equation by $-y_2$, multiply the second by $y_1$, and add the resulting equations, we obtain

$$(y_1y_2'' - y_1''y_2) + p(t)(y_1y_2' - y_1'y_2) = 0 \tag{9.17}$$

Next, we let $W(t) = W(y_1, y_2)(t)$ and observe that

$$W' = y_1y_2'' - y_1''y_2 \tag{9.18}$$

Then we can write Eq.(9.17) in the form

$$W' + p(t)W = 0. \tag{9.19}$$

Equation (9.19) can be solved immediately since it is both a first order linear equation and a separable equation. Thus

$$W(t) = c \exp\left[-\int p(t)dt\right], \tag{9.20}$$

where $c$ is a constant. The value of $c$ depends on which pair of solutions of Eq. (9.14) is involved. However, since the exponential function is never zero, $W(t)$ is not zero unless $c = 0$, in which case $W(t)$ is zero for all $t$, which completes the proof of the theorem.

***Example*** **10** Verify that the Wronskian of $y_1(t) = t^{1/2}$ and $y_2(t) = t^{-1}$ of solutions of the equation

$$2t^2y'' + 3ty' - y = 0, \quad t > 0 \tag{9.21}$$

is given by the equation

$$W(y_1, y_2)(t) = c \exp\left[-\int p(t)dt\right]$$

for some $c$.

*Solution*

It can be seen that

$$W(y_1, y_2)(t) = \begin{vmatrix} t^{\frac{1}{2}} & t^{-1} \\ \frac{1}{2}t^{-\frac{1}{2}} & -t^{-2} \end{vmatrix} = -\frac{3}{2}t^{-3/2}. \tag{9.22}$$

To use Eq. (9.15) in the Abel's theorem we must write the differential equation (9.21) in the standard form with the coefficient of

$y''$ equal to 1. Thus we obtain

$$y'' + \frac{3}{2t}y' - \frac{1}{2t^2}y = 0,$$

so $p(t) = 3/2t$. Hence, by Eq. (9.15),

$$W(y_1, y_2)(t) = c \exp\left[-\int \frac{3}{2t}dt\right] = c \exp\left(-\frac{3}{2}\ln t\right)$$

$$= c\,t^{-3/2}. \tag{9.23}$$

Equation (9.23) gives the Wronskian of any pair of solutions of Eq. (9.21). For the particular solutions given in this example we must choose $c = -3/2$, so as to make (9.22) and (9.23) equal.

**Exercises**

In each of Exercises 1-6, determine the longest interval in which the given initial value problem is certain to have a unique twice differentiable solution. Do not attempt to find the solution.

1. $ty'' + 3y = t, \quad y(2) = 1, \quad y'(2) = 3$
2. $(t-1)y'' - 3ty' + 4y = \sin t, \quad y(-2) = 2, \; y'(-2) = 1$
3. $t(t-4)y'' + 3ty' + 5y = 2, \; y(3) = 0, \; y'(3) = -1$
4. $y'' + (\cos t)y' + 3(\ln|t|)y = 0, \quad y(3) = 2, \quad y'(3) = 1$
5. $(x-3)y'' + xy' + (\ln|x|)y = 0, \quad y(1) = 0, \; y'(1) = 1$
6. $(x-2)y'' + y' + (x-2)(\tan x)y = 0, \; y(3) = 1, \; y'(3) = 2$

In each of Exercises 7-8, find the fundamental set of solutions specified by the Theorem 5 for the given differential equation and initial point.

7. $y'' + y' - 2y = 0, \quad t_0 = 0$

8. $y''+5y'+4y = 0, \quad t_0 = 1$ In each of Exercises 9-12, verify that the functions $y_1$ and $y_2$ are solutions of the given differential equation. Do they constitute a fundamental set of solutions?

9. $y'' + 4y = 0; \quad y_1(t) = \cos 2t, \quad y_2(t) = \sin 2t$

10. $y'' - 2y' + y = 0; \quad y_1(t) = e^t, \quad y_2(t) = te^t$

11. $x^2y''-x(x+2)y'+(x+2)y = 0, \quad x > 0; \quad y_1(x) = x, \quad y_2(x) = xe^x$

12. $(1 - x\cot x)y'' - xy' + y = 0, \quad 0 < x < \pi; \quad y_1(x) = x, \; y_2(x) = \sin x$

***Answers***

1. $0 < t < \infty$
3. $0 < t < 4$
5. $0 < x < 3$
2. $-\infty < t < 1$
4. $0 < t < \infty$
6. $2 < x < 3\pi/2$

7. $y_1(t) = \frac{1}{3}e^{-2t} + \frac{2}{3}e^t, y_2(t) = -\frac{1}{3}e^{-2t} + \frac{1}{3}e^t$

8. $y_1(t) = -\frac{1}{3}e^{-4(t-1)} + \frac{4}{3}e^{-(t-1)}, y_2(t) = -\frac{1}{3}e^{-4(t-1)} + \frac{1}{3}e^{-(t-1)}$

9. Yes

10. Yes

11. Yes

12. Yes

# Chapter 10

# Homogeneous Second Order Equations with Constant Coefficients

In this chapter we show how to solve the homogeneous linear equation of the form

$$ay'' + by' + cy = 0, \tag{10.1}$$

where $a$ and $b$ are constants.

## 10.1 Second Order Equations with Constant Coefficients

A differential equation of the form (10.1) is a homogeneous second order linear differential equation with constant coefficients. (We assume that $a, b$ and $c$ are real and the range of t considered is the entire t-axis.)

We know that the solution of the *first order* homogeneous linear differential equation with constant coefficients $y' + ky = 0$ is an exponential function, namely $y = ce^{-kt}$.

We thus infer that

$$y = e^{\lambda t} \tag{10.2}$$

might be a solution of the differential equation (10.1) if $\lambda$ is properly chosen. Substituting the function (10.2) and its derivatives $y' = \lambda e^{\lambda t}$ and $y'' = \lambda^2 e^{\lambda t}$ into the equation (10.1), we obtain

$$(a\lambda^2 + b\lambda + c)e^{\lambda t} = 0.$$

Hence the function (10.2) is a solution of the differential equation (10.1), if $\lambda$ is a solution of the quadratic equation

$$a\lambda^2 + b\lambda + c = 0 \tag{10.3}$$

The equation (10.3) is called the **characteristic equation** (or **auxiliary equation**) of (10.1).

Since $a$, $b$ and $c$ are real, from elementary algebra we know that, the characteristic equation may have

1. two distinct real roots if the discriminant $b^2 - 4ac > 0$.

2. two complex conjugate roots, if the discriminant $b^2 - 4ac < 0$.

3. a real double root. if the discriminant $b^2 - 4ac = 0$.

We consider these cases in detail.

### 10.1.1 Case 1 Two distinct real roots $\lambda_1$ and $\lambda_2$

In this case

$$y_1 = e^{\lambda_1 t}, \quad y_2 = e^{\lambda_2 t}$$

are solutions of the differential equation (10.1). The Wronskian of $y_1$ and $y_2$ is found out to be $(\lambda_2 - \lambda)\exp(\lambda_1 + \lambda_2)t$. When $\lambda_1$ and $\lambda_2$ are distinct, then wronskian is never 0 on any interval, so that $y_1$ and $y_2$ form a basis (fundamental system of solutions) on any interval. Hence the general solution is given by

$$y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \tag{10.4}$$

### 10.1.2 Case 2 Double Root

This case arises if and only if the discriminant of the characteristic equation (10.3) is zero, i.e. if $b^2 - 4ac = 0$. Then

$$\lambda_1 = \lambda_2 = -b/2a$$

The double root is $\lambda = -b/2a$, and we have at first only one solution $y_1 = e^{\lambda t}$ (where$\lambda = -b/2a$). To find another solution $y_2$, we may apply the *method of variation of parameters.*

The idea of method of variation of parameters is to start with $y(t) = v(t)y_1(t)$ where $y_1 = e^{\frac{-b}{2a}t}$

and to determine $v(t)$, so that $y_2(t)$ is a solution of (10.1).

We substitute $y(t)$ and its derivatives into (10.1) and collect terms, obtaining

$$v''y_1 = 0.$$

As $y_1$ is nonzero, this implies that $v'' = 0$. By two integrations, $v = c_1 t + c_2$. Hence

$$y(t) = uy_1 = (c_1 t + c_2)e^{\lambda t} = c_1 te^{\lambda t} + c_2 e^{\lambda t}.$$

The second term on the right side of the above equation corresponds to the original solution $y_1(t) = e^{\lambda t}$ , but the first term arises from a second solution, namely,$y_2(t) = te^{\lambda t}$. It can be verified that the Wronskian of $y_1$ and $y_2$ is never 0, so that

$$y_1(t) = e^{\lambda t}, y_2(t) = te^{\lambda t}$$

form a fundamental set of solutions.

The corresponding general solution is

$$y(t) = (c_1 + c_2 t)e^{\lambda t}, \left(\lambda = -\frac{b}{2a}\right)$$

**Attention**: If $\lambda$ is a *simple root* of (10.3), then the above is **not** a solution of (10.1).

### 10.1.3 Case 3 Complex Roots

When the discriminant$b^2 - 4ac < 0$, we obtain complex roots that occur in conjugate pairs, say $\lambda_1 = p + i\,q, \quad \lambda_2 = p - i\,q$ . Hence we first obtain the basis

$$y_1 = e^{(p+i\,q)t}, \quad y_2 = e^{(p-i\,q)t}$$

consisting of two *complex* functions. Practically one is interested in the real solutions obtainable from these complex solutions.

Now taking $\theta = qt$ in the Euler formulae

$$e^{i\,\theta} = \cos\theta + i\sin\theta$$

and $e^{-i\,\theta} = \cos\theta - i\sin\theta$,

we obtain

$$y_1 = e^{(p+i\,q)t} = e^{pt}e^{iqt} = e^{pt}(\cos qt + i\sin qt)$$

and

$$y_2 = e^{(p-i\,q)t} = e^{pt}e^{-iqt} = e^{pt}(\cos qt - i\sin qt).$$

Adding and subtracting the above two expressions for $y_1$ and $y_2$, we obtain

$$\frac{y_1 + y_2}{2} = e^{pt} \cos qt, \qquad \tfrac{y_1 - y_2}{2} = e^{pt} \sin qt \ .$$

The two functions on the right are real valued. Also, being the linear combinations of the solutions $y_1$ and$y_2$, both $e^{pt} \cos qt$ and $e^{pt} \sin qt$ are solutions of the differential equation (10.1). Since their quotient $\cos qt / \sin qt$ is not constant on any interval, they are linearly independent on any interval and hence form a fundamental set of solutions. Hence in the case of complex roots a *basis* on any interval is

$$e^{pt} \cos qt, e^{pt} \sin qt$$

and the corresponding general solution is

$$y(t) = e^{pt}(A \cos qt + B \sin qt), \tag{10.5}$$

where $A$ and $B$ are arbitrary constants.

**Working Rule: Determination of General Solution**

To find the general solution of the differential equation

$$ay'' + by' + cy = 0 \tag{10.6}$$

we first write the characteristic equation

$$a\lambda^2 + b\lambda + c = 0 \tag{10.7}$$

and then the general solution of (10.6) is given by the following table:

| *Case* | *Roots of the characteristic equation* $a\lambda^2 + b\lambda + c = 0$ | *Basis of* $ay'' + by' + cy = 0$ | *General solution of* $ay'' + by' + cy = 0$ |
|---|---|---|---|
| 1 | Distinct real roots $\lambda_1, \ \lambda_2$ | $e^{\lambda_1 t}, \quad e^{\lambda_2 t}$ | $y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$ |
| 2 | Real double root $\lambda = -\frac{b}{2a}$ | $e^{\lambda t}, \ t\, e^{\lambda t}$ | $y(t) = (c_1 + c_2 t)e^{\lambda\, t}$ |
| 3 | Complex conjugate roots $\lambda_1 = p + iq$, and $\lambda_2 = p - iq$ | $e^{pt}\cos qt$ $e^{pt}\sin qt$ | $y(t) = e^{pt}\,(A\cos qt + B\sin qt)$ |

***Example* 1** (D*istinct real roots*) Solve $y'' + y' - 2y = 0$.

*Solution*

The characteristic equation is $\lambda^2+\lambda-2 = 0$, that has the roots $\lambda_1 = 1$ and $\lambda_2 = -2$. The *basis* in this case is $e^t$ and $e^{-2t}$. Hence general solution is given by

$$y = y(t) = c_1 e^t + c_2 e^{-2t}.$$

***Example* 2** (*Distinct real roots*)

Find the solution of the initial value problem

$$y'' - y' - 2y = 0, \ y(0) = 0, \ y'(0) = 1.$$

*Solution*

The characteristic equation is

$$\lambda^2 - \lambda - 2 = 0.$$

The roots are $\lambda = -1$ and $\lambda = 2$. Hence a basis is

$e^{-t}$ and $e^{2t}$

and the general solution is

$$y = c_1 e^{-t} + c_2 e^{2t}.$$

Noting that

$$y' = -c_1 e^{-t} + 2c_2 e^{2t}$$

and using the initial conditions, we have

$c_1 + c_2 = 0$ and $-c_1 + 2c_2 = 1$.

Solving for $c_1$ and $c_2$, we obtain

$$c_1 = -\frac{1}{3}, \ c_2 = \frac{1}{3}.$$

Hence solution to the IVP is

$$y = -\frac{1}{3} e^{-t} + \frac{1}{3} e^{2t}.$$

***Example* 3** (*Double root*) Solve $y'' + 8y' + 16y = 0$.

*Solution*

The characteristic equation is $\lambda^2 + 8\lambda + 16 = 0$, and the double root $\lambda = -4$. The *basis* in this case is $e^{-4t}$ and $te^{-4t}$. Hence, general solution is given by

$$y = y(t) = (c_1 + c_2 t)e^{-4t}.$$

***Example* 4** (*Double root*) Find the solution of the initial value problem

$$y'' - 2y' + y = 0, y(0) = 1, \; y'(0) = 2$$

*Solution*

The characteristic equation is $\lambda^2 - 2\lambda + 1 = 0$ has the real double root $\lambda = 1$. The basis in this case is

$y_1 = e^t$ and $y_2 = te^t$.

and general solution is $y = (c_1 + c_2 t)e^t$. Noting that

$$y' = (c_1 + c_2 t)e^t + c_2 e^t,$$

the initial conditions give $c_1 = 1, \; c_2 = 1$.

Hence solution is

$$y = (1 + t)e^t.$$

***Example* 5** (*Complex conjugate roots*) Find a general solution of the equation $y'' + 10y' + 29y = 0$.

*Solution*

The characteristic equation is $\lambda^2 + 10\lambda + 29 = 0$, that has complex conjugate roots $\lambda_1 = p + i\,q = -5 + i\,2$ and $\lambda_2 = p - i\,q = -5 - i\,2$. Here $p = -5$ and $q = 2$. The *basis* in this case is $e^{-5t}\cos 2t$ and $e^{-5t}\sin 2t$. Hence general solution is given by

$$y = y(t) = e^{-5t}(A\cos 2t + B\sin 2t),$$

where $A$ and $B$ are arbitrary constants.

***Example* 6** (*Complex conjugate roots*) Find the solution of the initial value problem

$$y'' + \omega^2,\ y = 0y(0) = 1,\ y'(0) = \omega,$$

where $\omega$ is a nonzero constant.

*Solution*

The characteristic equation is $\lambda^2 + \omega^2 = 0$,and has complex conjugate roots $\lambda_1 = p + i\,q = i\,\omega$ and $\lambda_1 = p - i\,q = -i\,\omega$. Here $p = 0$ and $q = \omega$. The *basis in this case is* $\cos\omega t$ and $\sin\omega t$. Hence by (10.5), general solution is given by

$$y = y(t) = A\cos\omega t + B\sin\omega t,$$

where $A$ and $B$ are arbitrary constants. Noting that

$$y' = -A\omega\ \sin\ \omega t + B\omega\ \cos\ \omega t,$$

the initial conditions give $A = 1$ and $B = 1$.

Hence solution is

$$y = \cos \omega t + \sin \omega t.$$

### 10.1.4 Exercises

In Exercises 1-10, solve the following differential equations.

1. $y'' - 9y = 0$
2. $y'' + 9y = 0$
3. $y'' - 8y' + 16y = 0$
4. $y'' + 6y' + 9y = 0$
5. $y'' + y' + 0.25y = 0$
6. $y'' + 2y' = 0$
7. $8y'' - 2y' - y = 0$
8. $y'' + 2ky' + k^2 y = 0$
9. $2y'' + 10y' + 25y = 0$
10. $y'' + 2y' + (\omega^2 + 1)y = 0$

11. Applying the method of variation of parameters solve $y'' + 6y' + 9y = 0$, using $y_1 = e^{2t}$ as one of the solution. (Hint: Follow the method described in the Case 3 for double root)

12. Verify directly that in the case of a double root, $te^{\lambda t}$ with $\lambda = -\frac{a}{2}$ is a solution of $y'' + ay' + by = 0$.

13. Verify that if $y'' + ay' + by = 0$ does not have a double root, then $te^{\lambda t}$ is not a solution of it.

14. Assuming that $y = y(t) = e^{pt}(A \cos qt + B \sin qt)$ is a solution of $y'' + ay' + by = 0$, express $a$ and $b$ in terms of $p$ and $q$.

In Exercises 15-18, find a differential equation of the form $y'' + ay' + by = 0$ for which the given functions constitute a basis of solution.

15. $e^{-\alpha t}, \;\; te^{-\alpha t}$
16. $e^{-t}\cos\sqrt{5}t, \;\; e^{-t}\sin\sqrt{5}t$
17. $\cosh 6t, \;\; \sinh 6t$
18. $e^{-t/2}\cos 2t, \;\; e^{-t/2}\sin 2t$

19. In each of the following, find a differential equation of the form $y'' + ay' + by = 0$ for which the given $y$ is a general solution. Also determine the constants so that the given initial conditions are satisfied.

In Exercises 20- 25, solve the initial value problems.

20. $y'' - 16y = 0; \quad y(0) = 1, \;\; y'(0) = 20$
21. $y'' + 3y = 0; \quad y(0) = 2, \;\; y'(0) = 3\sqrt{3}$
22. $y'' - 4y' + 4y = 0; \quad y(0) = 0, \;\; y'(0) = -3$
23. $y'' + 4y' + (4 + \omega^2)y = 0; \quad y(0) = 1, \;\; y'(0) = \omega - 2$
24. $y'' + 6y' + 9y = 0; \quad y(0) = -4, \;\; y'(0) = 14$
25. $y'' + 2\alpha y' + (\alpha^2 + \pi^2)y = 0; \quad y(0) = 3, \;\; y'(0) = -3\alpha$

**Answers**

1. $y = c_1e^{3t} + c_2e^{-3t}$
2. $y = A\cos 3t + B\sin 3t$
3. $y = (c_1 + c_2t)e^{4t}$
4. $y = (c_1 + c_2t)e^{-3t}$

5. $y = (c_1 + c_2 t)e^{-t/2}$

6. $y = c_1 + c_2 e^{-2t}$

7. $y = c_1 e^{t/2} + c_2 e^{-t/4}$

8. $y = (c_1 + c_2 t)e^{-kt}$

9. $y = e^{-5t/2}(A \cos \frac{5}{2}t + B \sin \frac{5}{2}t)$

10. $y = e^{-t}(A \cos \omega t + B \sin \omega t)$

20. $y = 3e^{4t} - 2e^{-4t}$

22. $y = -3te^{2t}$

23. $y = e^{-2t}\,(\cos \omega t + \sin \omega t)$

24. $y = (2t - 4)e^{-3t}$

25. $y = 3e^{-at} \cos \pi t$

Chapter 11

# Solution by Reducing to First Order When One Solution is Known

Suppose that we know one solution$y_1(t)$, not everywhere zero, of

$$y'' + p(t)y' + q(t)y = 0 \tag{11.1}$$

To find a second solution, let

$$y = v(t)y_1(t); \tag{11.2}$$

then

$$y' = v'(t)y_1(t) + v(t){y_1}'(t)$$

and

$$y'' = v''(t)y_1(t) + 2v'(t){y_1}'(t) + v(t)y_1''(t).$$

Substituting for $y, y'$, and $y''$in Eq.(11.1), and collecting terms, we find that

$$y_1 v'' + (2y_1' + py_1)v' + (y_1'' + py_1' + qy_1)v = 0 \tag{11.3}$$

Since $y_1$ is a solution of Eq.(11.1), the coefficient of $v$ in Eq.(11.3) is zero, so that Eq.(11.3) becomes

$$y_1 v'' + (2{y_1}' + py_1)v' = 0. \tag{11.4}$$

Despite its appearance, Eq.(11.4) is actually a first order equation for the function $v'$ and can be solved either as a first order linear equation or as a separable equation. Once $v'$ has been found, then $v$ is obtained by integration. Finally, $y$ is determined from Eq.(11.2). This procedure is called the method of *reduction of order*, because the crucial step is the solution of a first order differential equation for $v'$ rather than the original second order equation for $y$. Although it is possible to write down a formula for $v(t)$, we will instead illustrate how this method works by an example.

***Example* 1** Given that $y_1(t) = t^{-1}$ is a solution of

$$2t^2 y'' + 3ty' - y = 0, \qquad t > 0 \tag{11.5}$$

by reducing to first order, find a fundamental set of solutions.

*Solution*

We set$y = v(t)t^{-1}$; then

$$y' = v't^{-1} - vt^{-2}, \quad y'' = v''t^{-1} - 2v't^{-2} + 2vt^{-3}.$$

Substituting for $y, y'$, and $y''$ in Eq.(11.5) and collecting terms, we obtain

$$2t^2(v''t^{-1} - vt^{-2} + 2vt^{-2} + 2vt^{-3}) + 3t(v't^{-1} - vt^{-2}) - vt^{-1}$$

$$\begin{aligned} &= 2tv'' + (-4+3)v't^{-2} + 2vt^{-3}) + 3t(v't^{-1} - vt^{-2}) - vt^{-1} \\ &= 2tv'' - v' = 0 \qquad (11.6) \end{aligned}$$

Note that the coefficient of $v$ is zero, as it should be; this provides a useful check on our algebra.

Separating the variables in Eq.(11.6) and solving for $v'(t)$, we find that

$$v'(t) = ct^{1/2}$$

then

$$v(t) = \frac{2}{3}ct^{3/2} + k.$$

It follows that

$$y = \frac{2}{3}ct^{1/2} + kt^{-1} \qquad (11.7)$$

where $c$ and $k$ are arbitrary constants. The second term on the right side of Eq. (11.7) is a multiple of $y_1(t)$and can be dropped,

but the first term provides a new solution$y_2(t) = t^{1/2}$.
It can be seen that Wronskian of $y_1$ and $y_2$ is

$$W(y_1, y_2)(t) = \frac{3}{2}t^{-3/2}, \quad t > 0 \tag{11.8}$$

Consequently, $y_1$ and $y_2$ form a fundamental set of solutions of the equation.

***Example* 2** Find a basis of solutions for the following second-order homogeneous linear equation for positive $t$ :

$$t^2y'' - ty' + y = 0. \tag{11.9}$$

*Solution*

By inspection a solution is$y_1 = t$, because $y_1' = 1$,and $y_1'' = 0$, substitution gives $-t \cdot 1 + t = 0$.
We set$y = v(t)t$; then

$$y' = v'(t)t + v(t);$$

$$y'' = v''(t)t + 2v'(t)$$

Substituting for $y, y'$ and $y''$ in Eq. (11.9) and collecting terms, we obtain

$$t^2[v''(t) + 2v'(t)] - t[v'(t)t + v(t)] + v(t)t = 0$$

i.e.,

$$t^3v''(t) + t^2v'(t) = 0$$

i.e.,

$$t\frac{d}{dt}(v'(t)) = -v'(t)$$

i.e.,

$$\frac{\frac{d}{dt}(v'(t))}{v'(t)} = -\frac{1}{t}.$$

Integrating,

$$\ln\, v'(t) = -\ln t + \ln c$$

where c is a arbitrary constant.

i.e.,

$$\ln\, v'(t) = \ln\, ct^{-1}$$

i.e.,

$$v'(t) = \frac{c}{t}$$

Integrating,

$$v(t) = c\ln\, t + k,$$

where $k$ is also an arbitrary constant.

Hence

$$y = ct\, \ln\, t + kt. \tag{11.10}$$

The second term on the right of (11.10) is a multiple of $y_1(t)$ and can be dropped, but the first term provides a new solution $y_2(t) = t\, \ln\, t$. It can be verified that the Wronskian of $y_1$ and $y_2$ is not equal to 0 for $t > 0$. Consequently $y_1$ and $y_2$ form a fundamental set of solution of Eq. (11.9).

### 11.0.5 Exercises

In Exercises 1 to 6, reduce to the first order and find a second solution of the given order differential equation (one solution is given)

1. $ty'' + 2y' + ty = 0, \quad y_1 = \frac{\sin t}{t}$

2. $t^2y'' - 5ty' + 9y = 0, \quad y_1 = t^3$

3. $t^2y'' + ty' + \left(t^2 - \frac{1}{4}\right) y = 0, \quad y_1 = t^{-1/2} \cos t$

4. $(1 - t^2)y'' - 2ty' + 2y = 0, \quad y_1 = t$

5. $t^2y'' + 2ty' - 2y = 0,\ t > 0;\ y_1(t) = t$

6. $t^2y'' - 4ty' + 6y = 0,\ t > 0;\ y_1(t) = t^2$

**Answers**

2. $y_2 = t^3 1\text{n } |t|$

3. $y_2 = t^{-1/2} \sin\ t$

5. $y_2 = t^{-2}$

6. $y_2 = t^3$

Chapter 12

# Euler-Cauchy Equation

**Cauchy equation** or **Euler equation** is a differential equation of the form

$$t^2y'' + aty' + by = 0 \tag{12.1}$$

where $a$ and $b$ are constants.

(12.1) can be solved by algebraic manipulations. By substituting,

$$y = t^m \tag{12.2}$$

and its derivatives into the differential equation (12.1), we find

$$t^2m(m-1)t^{m-2} + atmt^{m-1} + bt^m = 0.$$

By omitting $t^m$, which is not zero when $t \neq 0$, we obtain the auxiliary equation

$$m^2 + (a-1)m + b = 0. \tag{12.3}$$

**Case 1: Distinct Real Roots**

If the roots $m_1$ and $m_2$ of the equation (12.3) are different, then the two functions

$$y_1(t) = t^{m_1} \quad \text{and} \quad y_2(t) = t^{m_2}$$

constitute a basis of solutions of the differential equation (12.1) for all $t$ for which these functions are defined. The corresponding general solution is

$$y = y(t) = c_1 t^{m_1} + c_2 t^{m_2}, \tag{12.4}$$

where $c_1$, $c_2$ are arbitrary constants.

***Example* 1** (*Different roots*) Solve $t^2y'' + ty' - y = 0$.

*Solution*

Here $a = 1$, $b = -1$ and hence the auxiliary equation (12.3) becomes

$$m^2 - 1 = 0$$

having roots $m_1$=1 and $m_2$= $-1$.

Hence a basis of real solutions for all $t \neq 0$ is $y_1 = t$ and $y_2 = t^1$ and the corresponding general solution for all $t \neq 0$ is $y = c_1 t + c_2 t^1$.

***Example* 2** Solve $x^2y'' - 2.5xy' - 2y = 0$.

*Solution*

[Here independent variable is $x$ instead of $t$. Hence general solution will have the form $y = y(x) = c_1x^{m_1} + c_2x^{m_2}$.]

Here $a = 2.5,\; b = 2$ and hence the auxiliary equation (12.3) becomes

$$m^2 - 3.5m - 2 = 0$$

having roots $m_1 = 4$and $m_2 = -\frac{1}{2}$.
Hence a basis of real solutions for all $x \neq 0$ is $y_1 = x^4$ and $y_1 = \frac{1}{\sqrt{x}}$.
Hence the general solution for all $x \neq 0$ is

$$y_1 = c_1x^4 + c_2\frac{1}{\sqrt{x}}.$$

**Case 2 : Double Real Root**

The auxiliary equation (12.3) has a *double root* $m_1 = m_2$ if and only if $b = \frac{1}{4}(1-a)^2$, and then $m_1 = m_2 = \frac{1-a}{2}$. In this critical case we use the method of variation of parameters to find the second solution and obtain

$$y_2 = uy_1 = (\ln t)y_1.$$

Thus writing $m$ in place of $m_1$, we get

$$y_1 = t^m \quad \text{and} \quad y_2 = t^m \ln t, \tag{12.5}$$

(where $m = \frac{1-a}{2}$) as the solutions of (12.1) in the case of a double root $m$ of (12.3). It can be seen that these solutions are linearly independent, and hence they constitute a basis of real solutions of (12.1) for all positive $t$, and the corresponding general solution of (12.1) is

$$y = (c_1 + c_2 \ln t)t^m, \tag{12.6}$$

where $c_1,\ c_2$ are arbitrary.

***Example* 3** (*Double root*) Solve $x^2y'' - 3xy' + 4y = 0$.

*Solution*

Here $a = -3,\ b = 4$ and hence the auxiliary equation (12.3) becomes

$$m^2 - 4m + 4 = 0$$

which has the double root $m = 2$.

Hence a basis of real solutions for all positive $x$ is $x^2$ and $x^2 \ln x$, and the corresponding general solution is

$$y = (c_1 + c_2 \ln x)x^2$$

**Case 3: Complex Conjugate Roots**

If the roots of (12.3) are complex, then they occur in conjugate pair, say

$m_1 = \mu + i\,\nu$ and $m_2 = \mu - i\,\nu$.

Since $e^{\ln t} = t$, we note that $t^{i\,\nu}$ can be written as follows:

$$t^{i\,\nu} = \left(e^{\ln t}\right)^{i\,\nu} = e^{i\,\nu \ln t}$$

$= \cos(v \ln t) + i \sin(v \ln t)$, Using the Euler formula

$$e^{i\theta} = \cos\theta + i \sin\theta$$

Hence

$$t^{m_1} = t^{\mu} t^{i\,\nu} = t^{\mu}[\cos(v \ln t) + i \sin(v \ln t)] t^{m_2}$$

$$= t^{\mu} t^{-i\,\nu} = t^{\mu} e^{-i\,\nu \ln t} = t^{\mu}[\cos(v \ln t) - i \sin(v \ln t)].$$

Hence

$\frac{t^{m_1}+t^{m_2}}{2} = t^{\mu} \cos(v \ln t)$ and $\frac{t^{m_1}-t^{m_2}}{2i} = t^{\mu} \sin(v \ln t)$

and are the real solutions of (12.3) and hence the corresponding general solution for all positive $t$,

$$y = t^{\mu} \left[A \cos(v \ln t) + B \sin(v \ln t)\right] \tag{12.7}$$

***Example* 4** Solve $t^2 y'' + 7ty' + 13y = 0$.

*Solution*

Here $a = 7, \quad b = 13$, and hence the auxiliary equation (**??**) becomes

$$m^2 + 6m + 13 = 0$$

and gives

$$m = -3 \pm 2\,i .\,\because \qquad \mu = -3, \quad v = 2.$$

Hence by (12.7), the general solution of the given equation is

$$y = t^{-3} \left[A \cos(2 \ln t) + B \sin(2 \ln t)\right].$$

***Example* 5** (*General Solution in the case of a Double Root*)
The Euler-Cauchy equation

$$x^2y^n + 0.6xy' + 16.04y = 0$$

has the auxiliary equation $m^2 - 0.4m + 16.04 = 0$. The roots are complex conjugate, $m_1 = 0.2 + 4i$ and $m_2 = 0.2 - 4i$, where $i = \sqrt{-1}$. (We know from algebra that if a polynomial with real coefficients has complex roots, these are always conjugate.) Now use the trick of writing $x = e^{\ln\ x}$ and obtain

$$x^{m_1} = x^{0.2+4i} = x^{0.2}(e^{\ln\ x})^{4i} = x^{0.2}e^{(4\ \ln\ x)t},$$

$$x^{m_2} = x^{0.2-4i} = x^{0.2}(e^{\ln\ x})^{-4i} = x^{0.2}e^{-(4\ \ln\ x)t}.$$

Next apply Euler's formula with $i = 4\ \ln x$ to these two formulas. This gives

$$x^{m_1} = x^{0.2}[\cos(4\ln x) + i\sin(4\ln x)],$$

$$x^{m_2} = x^{0.2}[\cos(4\ln x) - i\ \sin(4\ln x)].$$

Add these two formulas, so that the sine drops out, and divide the result by 2. Then subtract the second formula from the first, so that the cosine drops out, and divide the result by $2i$. This yields

$x^{0.2}\cos\ (4\ln x)$ and $x^{0.2}\sin\ (4\ln x)$

respectively. By the superposition principle these are solutions of the Euler-Cauchy equation. Since their quotient $\cot(4\ln x)$ is not constant, they are linearly independent. Hence they form a basis of solutions, and the corresponding real general solution for all positive $x$ is

$$y = x^{0.2}[A\ \cos\,(4\ \ln\, x)\ + B\ \sin\,(4 \ln x)].$$

***Example* 6** Find the electrostatic potential $v = v(r)$ between two concentric spheres of radii $r_1 = 5\,\text{cm}$and $r_2 = 10\,\text{cm}$ kept at potentials $v_1 = 110V$ and $v_2 = 0$, respectively.

*Solution*

*Physical Information.* $v(r)$ is a solution of the Euler-Cauchy equation $rv'' + 2v' = 0$, where $v' = dv/dr$.

The auxiliary equation is $m^2 + m = 0$. It has the roots 0 and $-1$. This gives the general solution

$$v(r) = c_1 + c_2/r.$$

From the "boundary condition" (the potentials on the spheres) we obtain

$$v(5) = c_1 + \frac{c_2}{5} = 110, v(10) = c_1 + \frac{c_2}{10} = 0.$$

By subtraction, $\frac{c_2}{10} = 110,\ c_2 = 1100$. From the second equation, $c_1 = -\frac{c_2}{10} = -110$. Hence

$$v(r) = -110 + 1100/r\ \text{V}.$$

# Chapter 13

# Nonhomogeneous Equation

**Solution of Nonhomogeneous Differential Equations**

Consider the **non homogeneous linear equation**

$$L[y] = y'' + p(t)y' + q(t)y = g(t) \tag{13.1}$$

where $p$, $q$, and $g$ are given (continuous) functions on the open interval $I$. The equation

$$L[y] = y'' + p(t)y' + q(t)y = 0, \tag{13.2}$$

in which $g(t) = 0$ and $p$ and $q$ are the same as in Eq.(13.1), is called the **homogeneous equation** corresponding to Eq.(13.1). The following two results describe the structure of solutions of the nonhomogeneous equation (13.1) and provide a basis for constructing its general solution.

**Theorem 1** If $Y_1$ and $Y_2$ are two solutions of the non homogeneous equation (13.1), then their difference $Y_1 - Y_2$ is a solution of the corresponding homogeneous equation (13.2).

If, in addition, $y_1$ and $y_2$ are a fundamental set of solutions of Eq. (13.2), then

$$Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t), \tag{13.3}$$

where $c_1$ and $c_2$ are certain constants.

**Theorem 2** The general solution of the non homogeneous equation (13.1) can be written in the form

$$y = \phi(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t),$$

where $y_1$ and $y_2$ are a fundamental set of solutions of the corresponding homogeneous equation (13.2), $c_1$ and $c_2$ are arbitrary constants, and $y_p$ is some particular solution of the non homogeneous equation (13.1). That is, the **general solution** to (13.1) has the form

$$y = y_h + y_p \tag{13.4}$$

where $y_h$, called **complementary function,** is the general solution of the corresponding homogeneous equation (13.2) and $y_p$ is the **particular solution** of equation (13.1) which contains no arbitrary constant.

We are already familiar with methods of finding $y_h$ (i.e., general solution of homogeneous equations). For determining the particular solution $y_p$ various methods exist. In the coming chapters we discuss three methods: (a) method of undetermined coefficients, (b) method of variation of parameters (c), and series solution.

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Chapter 14

# Method of Undetermined Coefficients

## 14.1 Method of Undetermined Coefficients

**Method of undetermined coefficients** is a simpler method for determining the particular solution $y_p$ of second order nonhomogeneous differential equations of the form

$$ay'' + by' + cy = g(t) \tag{14.1}$$

with constant coefficients. The disadvantage of this method is that it is applicable only to those nonhomogeneous linear differential equations whose right hand side $g(t)$ is a single power of $t$, a polynomial, an exponential function, a sine or cosine or a sum or a product of such functions.

The Method of undetermined coefficients is based upon substituting the *general form* of the particular integral $y_p$ into the differential equation (14.1) and then determining the specific form by forcing the assumed particular integral to be a solution.

**Working Rule for the Determination of General Solution of Non-Homogeneous Equation**

To determine the general solution of

$$ay'' + by' + cy = g(t) \tag{14.1}$$

we proceed as follows:

**Step1:** Find the solution $y_h$ of the corresponding homogeneous equation

$$ay'' + by' + cy = 0 \tag{14.2}$$

**Step 2:** Find the particular solution $y_p$ of the non- homogeneous equation using Rule 1 or Rule 2 (discussed below.)

**Step 3:** Then general solution of (14.1) is $y = y_h + y_p$.

**RULE 1 (Basic Rule)**

If $g(t)$ in

$$ay'' + by' + cy = r(t)$$

is one of the functions in column 1 of the Table below, *choose* the corresponding function $y_p$ in column 2 and determine its undetermined coefficients by substituting $y_p$ into (14.1).

**Table** *(Method of undetermined coefficients RULE 1)*

In the following table we assume that

$$P_n(t) = a_n t^n + a_{n-1} t^{n-1} + \ \cdots \ + \ a_1 t + a_0.$$

| *Column* 1 | *Column* 2 |
|---|---|
| **Term in $g(t)$** | **Choice for $y_p$** |
| $ke^{pt}$ | $Ce^{pt}$ |
| $P_n(t) \quad (n = 0,\ 1,\ \ldots)$ | $K_n t^n + K_{n-1} t^{n-1} + \ \cdots \ + \ K_1 t + K_0$ |
| $k \cos qt$ | $K \cos qt + M \sin qt$ |
| $k \sin qt$ | $K \cos qt + M \sin qt$ |
| $k\, e^{pt} \cos qt$ | $e^{pt}\,(K \cos qt + M \sin qt)$ |
| $k\, e^{pt} \sin qt$ | $e^{pt}\,(K \cos qt + M \sin qt)$ |
| $P_n(t) e^{pt}$ | $\left(K_n t^n + K_{n-1} t^{n-1} + \ \cdots \ + \ K_1 t + K_0\right) e^{pt}$ |
| $P_n(t) e^{pt} \cos qt$ | $(K_n t^n + K_{n-1} t^{n-1} + \ \cdots \ + K_0) e^{pt} \cos qt$ <br> $+(L_n t^n + L_{n-1} t^{n-1} + \ \cdots \ + L_0) e^{pt} \sin qt$ |
| $P_n(t) e^{pt} \sin qt$ | $(K_n t^n + K_{n-1} t^{n-1} + \ \cdots \ + K_0) e^{pt} \cos qt$ <br> $+(L_n t^n + L_{n-1} t^{n-1} + \ \cdots \ + L_0) e^{pt} \sin qt$ |

***Example* 1** (*Using Rule* 1) Solve the nonhomogeneous equation

$$y'' - 4y' + 3y = 10e^{-2t}. \tag{14.3}$$

*Solution*

**Step 1:** It can be found out that the general solution to the corresponding homogeneous equation

$$y'' - 4y' + 3y = 0$$

is

$$y_h = c_1 e^t + c_2 e^{3t}.$$

**Step 2***:* Determination of $y_p$ :

Here $g(t)$ is $10e^{-2t}$, hence by Rule 1, the *choice* for $y_p$ is $Ce^{-2t}$. *Substituting* this into the given nonhomogeneous equation we get

$$y_p'' - 4y_p' + 3y_p = 10e^{-2t}$$

i.e.,

$$4Ce^{-2t} - 4(-2Ce^{-2t}) + 3Ce^{-2t} = 10e^{-2t}.$$

Hence

$$4C + 8C + 3C = 10$$

so that $C = \frac{2}{3}$.

Therefore $y_p = \frac{2}{3}e^{-2t}$.

**Step 3:** The general solution to the nonhomogeneous equation is

$$y = y_h + y_p = c_1 e^t + c_2 e^{3t} + \frac{2}{3}e^{-2t}.$$

***Example* 2** (*Using Rule* 1) Solve the nonhomogeneous equation

$$y'' + 4y = 8\,t^2.$$

*Solution*

**Step 1:** It can be found out that the general solution to the corresponding homogeneous equation $y'' + 4y = 0$ is

$$y_h = A\cos 2t + B\sin 2t.$$

**Step 2:** Determination of $y_p$ :

Here $g(t)$ is $8t^2$, hence by Rule 1, the *choice* for $y_p$ is

$$y_p = K_2t^2 + K_1t + K_0.$$

Substituting this into the given nonhomogeneous equation,t

$$y_p'' + 4y_p = 8\,t^2.$$

i.e. $2K_2 + 4(K_2x^2 + K_1x + K_0) = 8\,t^2$.

Equating the coefficients of like powers of $t$, we get

$$4K_2 = 8, \qquad 4K_1 = 0, \qquad 2K_2 + 4K_0 = 0,$$

which gives

$$K_2 = 2, \qquad K_1 = 0, \qquad K_2 = -1.$$

Hence

$$y_p = 2t^2 - 1.$$

**Step 3:** The general solution to the nonhomogeneous equation is

$$y = y_h + y_p = A\cos 2t + B\sin 2t + 2t^2 - 1.$$

***Example* 3** (*Using Rule* 1) Solve the nonhomogeneous equation

$$y'' - y' - 2y = 10\cos t.$$

*Solution*

**Step 1:** It can be found out that the general solution to the corresponding homogeneous equation $y'' - y' - 2y = 0$ is

$$y_h = c_1 e^{-t} + c_2 e^{2t}.$$

**Step 2:** Determination of $y_p$ :

Here $g(x)$is $10\cos x$, hence by Rule 1, the *choice* for $y_p$ is

$$y_p = K\cos t + M\sin t.$$

*Substituting* this into the given nonhomogeneous equation,

$$y_p'' - y_p' - 2y_p = 10\cos t.$$

i.e., $(-3K - M)$ cos $x$ + $(K - 3M)$ sin $x$ = 10 cos $x$.

By equating the coefficients of cos $x$ and sin $x$ on both sides,

$-3K - M = 10$ and $K - 3M = 0$,

which gives $K = -3$ and $M = -1$, so that

$$y_p = -3\cos t - \sin t.$$

**Step 3:** The general solution to the nonhomogeneous equation is

$$y = y_h + y_p = c_1 e^{-t} + c_2 e^{2t} - 3\cos t - \sin t.$$

***Example* 4** Find a particular solution of

$$y'' - 3y' - 4y = 3e^{2t}. \tag{14.4}$$

*Solution*

The choice for $y_p$ is

$$y_p(t) = C\, e^{2t},$$

where $C$ is a constant to be determined. Hence

$$y_p'(t) = 2C\, e^{2t}, y_p''(t) = 4C\, e^{2t},$$

and substituting for $y$, $y'$, and $y''$ in Eq.(14.4), we obtain

$$(4C - 6C - 4C)e^{2t} = 3e^{2t}.$$

Hence, $-6Ce^{2t}$ must equal to $3e^{2t}$, so $C = -\frac{1}{2}$. Thus a particular solution is

$$y_p(t) = -\frac{1}{2}e^{2t}.$$

***Example* 5** Find a particular solution of

$$y'' - 3y' - 4y = 2\sin t \tag{14.5}$$

*Solution*

The choice for $y_p$ is

$$y_p(t) = A\sin t + B\cos t,$$

where $A$ and $B$ are to be determined. Then

$$y_p'(t) = A\cos t + B\sin t, y_p''(t) = -A\sin t - B\cos t, .$$

By substituting these expressions for $y, y'$, and $y''$ in Eq. (14.5) and collecting terms, we obtain

$$(-A + 3B - 4A)\sin\ t + (-B - 3A - 4B)\cos t = 2\sin t. \quad (14.6)$$

To satisfy Eq.(14.6), we must match the coefficients of $\sin t$ and $\cos t$ on each side of the equation; thus A and B must satisfy the equations

$$-5A + 3B = 2, -3A - 5B = 0.$$

Hence $A = -5/17$ and $B = 3/17$, so a particular solution of Eq.(14.5) is

$$y_p(t) = -\frac{5}{17}\sin t + \frac{3}{17}\cos t.$$

***Example* 6** Find a particular solution of

$$y'' - 3y' - 4y = -8e^t\cos 2t \quad (14.7)$$

*Solution*

The choice for $y_p$ is

$$y_p(t) = Ae^t \cos 2t + Be^t \sin 2t.$$

It follows that

$$y_p'(t) = (A + 2B)e^t \cos 2t + (-2A + B)e^t \sin 2t$$

and

$$y_p''(t) = (-3A + 4B)e^t \cos 2t + (-4A - 3B)e^t \sin 2t$$

By substituting these expressions in Eq.(14.7), we find that $A$ and $B$ must satisfy

$$10A + 2B = 8, \quad 2A - 10B = 0$$

Hence $A = 10/13$ and$B = 2/13$; therefore a particular solution of Eq.(14.7) is

$$y_p(t) = \frac{10}{13}e^t \cos 2t + \frac{2}{13}e^t \sin 2t..$$

**RULE 1B (Sum Rule)**

If $g(t)$ in

$$ay'' + by' + cy = g(t) \tag{14.8}$$

is a sum of functions in several lines of column 1 of the Table given above, *choose* for $y_p$ the sum of functions in the corresponding lines in column 2 and determine its undetermined coefficients by substituting $y_p$ into (14.8). The following theorem is helpful in this regard.

**Theorem (Principle of Superposition)** If $y_1(t)$ is a particular solution of

$$y'' + ay' + by = g_1(t)$$

and $y_2(t)$ is a particular solution of

$$y'' + ay' + by = g_2(t),$$

then

$$y(t) = y_1(t) + y_2(t)$$

is a solution of

$$y'' + ay' + by = g_1(t) + g_2(t)..$$

***Example* 7** (*Using Rule* 1B/Principle of Superposition) Find a particular solution of

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^t \cos 2t \tag{14.9}$$

*Solution*

By splitting up the right side of Eq.(14.9), we obtain the three differential equations

$$y'' - 3y' - 4y = 3e^{2t},$$

$$y'' - 3y' - 4y = 2\sin t,$$

and

$$y'' - 3y' - 4y = -8e^t \cos 2t.$$

Solutions of these three equations have been found in Examples 4,5, and 6 respectively.

Therefore a particular solution of Eq. (14.9) is their sum, and is

$$y_p(t) = -\frac{1}{2}e^{2t} + \frac{3}{17}\cos t - \frac{5}{17}\sin t + \frac{10}{13}e^t \cos 2t + \frac{2}{13}e^t \sin 2t.$$

***Example* 8** (*Using Rule* 1B) Solve the nonhomogeneous equation

$$y'' - 3y' + 2y = 4t + e^{3t}.$$

*Solution*

**Step 1:** It can be found out that the general solution to the corresponding homogeneous equation $y'' - 3y' + 2y = 0$ is

$$y_h = c_1 e^t + c_2 e^{2t}.$$

**Step 2:** Determination of $y_p$ : Here $g(t)$ is $4t + e^{3t}$, hence by Rule 1B, the *choice* for $y_p$ is

$$y_p = K_1 t + K_0 + Ce^{3t}.$$

Substituting this into the given nonhomogeneous equation and solving, we get

$$K_1 = 2, \qquad K_0 = 3, \qquad C = \frac{1}{2}$$

so that $y_p = 2t + 3 + \frac{1}{2}e^{3t}$.

**Step 3:** The general solution to the nonhomogeneous equation is

$$y = y_h + y_p = c_1 e^t + c_2 e^{2t} + 2t + 3 + \frac{1}{2}e^{3t}.$$

***Example* 9** (*Using Rule* 1B) Solve the nonhomogeneous equation

$$(D^2 - 2D + 3)y = t^3 + \sin t,$$

where $D$ is the differential operator given by $D \equiv \frac{d}{dt}$.

*Solution*

**Step 1:** It can be found out that the general solution to the corresponding homogeneous equation $(D^2 - 2D + 3)y = 0$ is

$$y_h = e^t(A\cos\sqrt{2}t + B\sin\sqrt{2}t).$$

**Step 2:** Determination of $y_p$ : Here $g(t)$ is $t^3 + \sin t$, hence the *choice* for $y_p$ is

$$y_p = K_3t^3 + K_2t^2 + K_1t + K_0 + K\cos t + M\sin t.$$

Substituting this into the given nonhomogeneous equation and solving,

$$K_3 = \frac{9}{27}, \quad K_2 = \frac{18}{27}, \quad K_1 = \frac{6}{27}, \quad K_0 = -\frac{8}{27}, \quad K = \frac{1}{4}, \quad M = \frac{1}{4},$$

so that

$$y_p = \frac{1}{27}(9t^3 + 18t^2 + 6t - 8) + \frac{1}{4}(\cos t + \sin t).$$

**Step 3:** The general solution to the nonhomogeneous equation is

$$y = y_h + y_p = e^t(A\cos\sqrt{2}t + B\sin\sqrt{2}t)$$

$$+\frac{1}{27}(9t^3 + 18t^2 + 6t - 8) + \frac{1}{4}(\cos t + \sin t).$$

***Example* 10** Solve the nonhomogeneous equation

$$y'' - 4y' + 3y = \sin 3x \cos 2x.$$

*Solution*

We note that the given differential equation can be written as

$$y'' - 4y' + 3y = \frac{1}{2}\left(\sin 5x + \sin x\right).$$

**Step 1:** The general solution to the corresponding homogeneous equation $y'' - 4y' + 3y = 0$ is

$$y_h = c_1e^{3x} + c_2e^x.$$

**Step 2:** Determination of $y_p$ :

Here $r(x) = \frac{1}{2}(\sin 5x + \sin x)$. Hence the *choice* for $y_p$ is

$$y_p = K_1 \cos 5x + M_1 \sin 5x + K_2 \cos x + M_2 \sin x$$

Substituting this into the given nonhomogeneous equation and solving, we get

$$K_1 = \frac{10}{884}, \quad M_1 = -\frac{11}{884}, \quad K_2 = \frac{1}{10}, \quad M_2 = \frac{1}{20}.$$

Hence

$$y_p = \frac{10}{884}\cos 5x - \frac{11}{884}\sin 5x + \frac{1}{10}\cos x + \frac{1}{20}\sin x.$$

**Step 3:** The general solution to the nonhomogeneous equation is

$$c_1 e^x + c_2 e^{3x} + \frac{10}{884}\cos 5x - \frac{11}{884}\sin 5x + \frac{1}{10}\cos x + \frac{1}{20}\sin x$$

***Attention:** If a term in $g(t)$ is a solution of the homogeneous equation corresponding to

$$ay'' + by' + cy = g(t),$$

we have to modify the choice of $y_p$ . We first consider an example.

***Example* 11** Find a particular solution of

$$y'' - 3y' - 4y = 2e^{-t} \tag{14.10}$$

*Solution*

We assume that choice for particular solution is $y_p(t) = Ae^{-t}$. By substituting in Eq.(14.10), we obtain

$$(A + 3A - 4A)e^{-t} = 2e^{-t} \tag{14.11}$$

i.e.,

$$0 = 2e^{-t} \tag{14.12}$$

Since the left side of Eq.(14.12) is zero, there is no choice of $A$ that satisfies this equation. Therefore, there is no particular solution of Eq.(14.10) of the assumed form. The reason for this possibly unexpected result becomes clear if we solve the homogeneous equation

$$y'' - 3y' - 4y = 0 \tag{14.13}$$

that corresponds to Eq.(14.10). A fundamental set of solutions of Eq.(14.13) is $y_1(t) = e^{-t}$ and $y_2(t) = e^{4t}$. Thus our assumed particular solution of Eq.(14.10) is actually a solution of the homogeneous equation (14.13); consequently, it cannot possibly be a solution of the non-homogeneous equation (14.10). To find a solution of Eq.(14.10), we must therefore consider functions of a somewhat different form.

At this stage, we look for a first order equation analogous to Eq.(14.10). One possibility is the linear equation

$$y' + y = 2e^{-t}. \tag{14.14}$$

If we try to find a particular solution of Eq.(14.14) of the form $Ae^{-t}$, we will fail because $e^{-t}$ is a solution of the homogeneous equation $y' + y = 0$. However, to solve Eq.(14.14) we proceed as follows: An integrating factor is $\mu(t) = e^t$, and by multiplying by $\mu(t)$ and then integrating both sides, we obtain the solution

$$y = 2te^{-t} + ce^{-t}. \tag{14.15}$$

The second term on the right side of Eq.(14.15) is the general solution of the homogeneous equation $y' + y = 0$, but the first term is a solution of the full nonhomogeneous equation (14.14). Observe that it involves the exponential factor $e^{-t}$ multiplied by the factor $t$. This is the clue that we were looking for.

We now return to Eq.(14.11) and assume a particular solution of the form $Y(t) = Ate^{-t}$. Then

$$Y'(t) = Ae^{-t} - Ate^{-t}, \qquad Y''(t) = -2Ae^{-t} + Ate^{-t} \tag{14.16}$$

Substituting these expressions for $y$, $y'$, and $y''$ in Eq., we obtain

$$(-2A - 3A)e^{-t} + (A + 3A - 4A)te^{-t} = 2e^{-t}.$$

Hence $-5A = 2$, so $A = -\frac{2}{5}$. Thus a particular solution is

$$Y(t) = -\frac{2}{5}te^{-t}.$$

The method discussed in the above example suggests the following modification rule.

**RULE 2** (**Modification rule**) If $g(t)$ in

$$ay'' + by' + cy = g(t) \tag{14.17}$$

is one of the functions in column 1 of the Table and *also* a solution of the homogeneous equation corresponding to (14.17), the *choice* for $y_p$ is as follows:

Multiply the expression in the appropriate line of column 2 by $t$ if this solution corresponds to a simple root of the characteristic equation of the corresponding homogeneous equation or by $t^2$ if this solution corresponds to a double root of the characteristic equation of the corresponding homogeneous equation.

***Example* 12** (*Using Rule* 2 (*Modification Rule*)) Solve the initial value problem

$$y'' - y' - 2y = 3e^{2t}, \qquad y(0) = 0, \quad y'(0) = -2.$$

*Solution*

**Step 1:** $y_h = c_1 e^{-t} + c_2 e^{2t}$

**Step 2:** Since $e^{2t}$ appears in $g(t)$ and$y_h$, we have to apply Rule 2. Since 2 is a simple root of the characteristic equation of the corresponding homogeneous equation, we have to multiply $e^{2t}$ by$t$. Hence a choice for $y_p$ is

$$y_p = C\,te^{2t}.$$

Substituting this into the non-homogeneous equation,

$$y_p'' - y_p' - 2y_p = 3e^{2t}$$

and we get

$$\left(Ct\,e^t\right)'' - \left(C\,t\,e^t\right)' - 2\left(C\,t\,e^t\right) = 3e^{2t},$$

which on simplification and solving for $C$ gives $C = 1$.

Hence $y_p = \, t\,e^t$

**Step 3** Hence the general solution is

$$y = y(x) = y_h + y_p = c_1e^{-t} + c_2e^{2t} + te^{2t}.$$

Now to find the particular solution satisfying the given initial conditions:

$$y(0) = 0 \quad \Rightarrow \quad c_1 + c_2 = 0. \Rightarrow c_1 = -c_2.$$

Also, $y' = y'(t) = -c_1e^{-t} + 2c_2e^{2t} + e^{2t} + 2te^{2t}$ and hence

$$y'(0) = -2 \qquad \Rightarrow \qquad -c_1 + 2c_2 + 1 = -2 \Rightarrow 3c_2 = -3 \qquad \Rightarrow c_2 = -1.$$

Hence, $c_1 = -\,c_2 = -1$, and the general solution is given by

$$y = e^{-t} - e^{2t} + te^{2t}.$$

***Example* 13** Find a particular solution of

$$y'' + y = \sin t \tag{14.18}$$

*Solution*

The reduced homogeneous equation $y'' + y = 0$ has

$$y = c_1 \sin t + c_2 \cos t$$

as its general solution, so it is useless to take $y_p = A \sin t + B \cos t$as a trial for the particular solution. Our choice is

$$y_p = t(A \sin t + B \cos t).$$

This yields

$$y_p' = A \sin t + B \cos t + t(A \cos t - B \sin t)$$

and

$$y_p'' = 2A \cos t - 2B \sin t + t(-A \sin t - B \cos t)$$

and by substituting in (14.18) we obtain

$$2A \cos t - 2B \sin t = \sin t.$$

Hence

$A = 0$ and $B - \frac{1}{2}$

satisfies our requirement, so

$$y_p = -\tfrac{1}{2}t\,\cos t$$

is the desired particular solution.

**RULE 2B (Modification and Sum Rules)**

If $g(t)$ in

$$ay'' + by' + cy = g(t) \tag{14.19}$$

is a sum of functions in column 1 of the Table, and if at least one of the functions is *also* a solution of the homogeneous equation corresponding to (14.19), we use modification and sum rules for the *choice* for$y_p$. This is illustrated in the following example.

***Example* 14** (*Using Rule* 2B) Solve the nonhomogeneous equation

$$y'' - 2y' + y = (D-1)^2 y = t + e^t.$$

*Solution*

**Step 1:** It can be found out that the general solution to the corresponding homogeneous equation $y'' - 2y' + y = 0$ is

$$y_h = (c_1 + c_2 t)e^t.$$

**Step 2:** Determination of $y_p$:

Here $g(t)$ is $t + e^t$. Note that $e^t$ is a solution of the homogeneous equation. Hence we have to apply Rule 2.

1 is *double root* of the characteristic equation

$$(\lambda - 1)^2 = 0.$$

Hence by the modification rule the term $e^t$ call for the particular solution $Ct^2e^t$, instead of $Ce^t$. Hence the *choice* for $y_p$ is

$$y_p = K_1 t + K_0 + Ct^2 e^t.$$

Substituting this into the given nonhomogeneous equation and solving,

$$K_1 = 1, \qquad K_0 = 2, \qquad C = \frac{1}{2},$$

so that

$$y_p = t + 2 + \frac{1}{2}t^2 e^t.$$

**Step 3:** The general solution to the nonhomogeneous equation is

$$y = y_h + y_p = (c_1 + c_2 t)e^t + t + 2 + \frac{1}{2}t^2 e^t.$$

***Example* 15** (*Using Rule 2B*) Solve the non homogeneous equation

$$y'' - 4y' + 4y = (D - 2)^2 y = t^3 e^{2t} + te^{2t}.$$

*Solution*

**Step 1:** The general solution to the corresponding homogeneous equation $y'' - 4y' + 4y = 0$ is

$$y_h = (c_1 + c_2 t)e^{2t}.$$

**Step 2:** Determination of $y_p$ :

Here $g(t)$ is $t^3e^{2t} + te^{2t}$. Note that $e^{2t}$ is a solution of the homogeneous equation. Hence we have to apply Rule 2. 2 is the *double root* of the characteristic equation

$$(\lambda - 2)^2 y = 0.$$

Hence by Rule 2, the term $t^3e^{2t}$ call for the particular solution $t^2 \cdot t^5 e^{2t} = t^5 e^{2t}$ and its linearly independent derivatives (not contained in $g(t)$). Hence the *choice* for $y_p$ is

$$y_p = K_3 t^5 e^{2t} + K_2 t^4 e^{2t} + K_1 t^3 e^{2t} + K_0 t^2 e^{2t}.$$

(Note that terms involving $te^{2t}$ and $e^{2t}$ are not included, since they appear in$y_h$.) Substituting this into the given nonhomogeneous equation and solving, we get

$$K_3 = \frac{1}{20}, \quad K_2 = 0, \quad K_1 = \frac{1}{6}, \quad K_0 = 0,$$

so that

$$y_p = \frac{1}{20} t^5 e^{2t} + \frac{1}{6} t^3 e^{2t}.$$

**Step 3:** The general solution to the nonhomogeneous equation is

$$y = (c_1 + c_2 t)e^{2t} + \frac{1}{20} t^5 e^{2t} + \frac{1}{6} t^3 e^{2t}.$$

**Exercises Set A**

In Exercises 1-7, find a particular solution of the differential equations. Also give the general solution. [*Note*: In the following example, we have the independent variable $x$ instead of the variable $t$].

1. $y'' + y = 3x^2$
2. $y'' - y' - 2y = 6e^x$
3. $y'' + 2y' + 3y = 27x$
4. $y'' + y = 6\sin x$
5. $y'' + 4y' + 4y = 18\cosh x$
6. $(D^2 - 1)y = 2x^2$
7. $(D^2 - 5D + 4)y = 10\cos x$

In Exercises 8-21, find a general solution of the following differential equations:

8. $y'' + y = x^2 + x$
9. $y'' + 3y' + 2y = x^3 + x$
10. $y'' + 3y' + y = 3e^x$
11. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 10e^{-2x}$
12. $y'' - 2y' + y = e^x$
13. $y'' - y' - 2y = 6e^x$
14. $(D^2 - 1)y = 2x^2$
15. $y'' + 2y' + y = \cos x$
16. $(D^2 + 1)y = 10e^x \sin x$
17. $y'' - 3y' + 2y = 4x + e^{3x}$
18. $(D^2 + 1)y = \sin x$
19. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = x^2 + e^{4x}$.
20. $\left(D^2 + 4\right) y = \sin 2x$.
21. $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = \sin 3x$.

In Exercises 22-24, solve the initial value problems.

22. $y'' - y' - 2y = 3e^{2x}, \quad y(0) = 0, \; y'(0) = -2$

23. $y'' - 4y' + 3y = 4e^{3x}, \quad y(0) = -1, \; y'(0) = 3$

24. $y'' + y' - 2y = -6\sin 2x - 18\cos 2x, \quad y(0) = 2, \; y'(0) = 2.$

    **Hint to Exercise* 5: $\cosh x = \frac{e^x + e^{-x}}{2}$.

***Answers***

1. $3x^2 - 6, \quad y = A\cos x + B\sin x + 3x^2 - 6$

2. $-3e^x$

3. $9x - 6$

4. $-3x\cos x, \quad y = A\cos x + B\sin x - 3x\cos x$

5. $e^x + 9e^{-x}, \quad y = (c_1 + c_2 x)\, e^{-2x} + e^x + 9e^{-x}$

6. $-2x^2 - 4, \quad y = c_1 e^x + c_2 e^{-x} - 2x^2 - 4$

7. $\frac{15}{17}\cos x - \frac{25}{17}\sin x, y = c_1 e^x + c_2 e^{4x}\frac{15}{17}\cos x - \frac{25}{17}\sin x$

8. $A\cos x + B\sin x + x^2 + x - 2$

9. $c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{8}(4x^3 - 18x^2 + 46x - 51)$

10. $c_1 e^{\frac{-3-\sqrt{5}}{2}x} + c_2 e^{\frac{-3+\sqrt{5}}{2}x} + \frac{3}{5}e^x$

11. $c_1 e^x + c_2 e^{3x} + \frac{2}{3}e^{-2x}$

12. $(c_1 + c_2 x)e^x + \frac{1}{2}x^2 e^x$

13. $c_1 e^{-x} + c_2 e^{2x} - 3e^x$

14. $c_1 e^{-x} + c_2 e^{2x} + 2x^2 + 4$

15. $(c_1 + c_2x)e^x + \frac{1}{2}\sin x$

16. $A\cos x + B\sin x + 2e^x(\sin x - 2\cos x)$

17. $c_1e^x + c_2e^{2x} + 2x + 3 + \frac{1}{2}e^{3x}$

18. $A\cos x + B\sin x - \frac{x}{2}\cos x$

19. $c_1e^x + c_2e^{2x} + \frac{e^{4x}}{6} + \frac{x^2}{2} + \frac{3x}{4} + \frac{7}{4}$

20. $A\cos 2x + B\sin 2x - \frac{x}{4}\cos 2x$

21. $y = c_1e^x + c_2e^{3x} + \frac{1}{234}\left(15\cos 3x - 3\sin 3x\right).$

22. $e^{-x} - e^{2x} + xe^{2x}$

23. $e^{3x} - 2e^x + 2xe^{3x}$

24. $3\cos 2x - e^{-2x}$

**Exercises Set B**

In Exercises 1-11, find the general solution of each of the following equations:

1. $y'' + 3y' - 10y = 6e^{4x};$

2. $y + 4y = 3\sin x;$

3. $y'' + 10y' + 25y = 14e^{-5x};$

4. $y'' - 2y' + 5y = 25x^2 + 12;$

5. $y'' - y' - 6y = 20e^{-2x};$

6. $y'' - 3y' + 2y = 14\sin 2x - 18\cos 2x$;

7. $y'' + y = 2\cos x$;

8. $y'' - 2y' = 12x - 10$;

9. $y'' - 2y' + y = 6e^x$;

10. $y'' - 2y' + 2y = e^x \sin x$

11. $y'' + y' = 10x^4 + 2$.

12. If $k$ and $b$ are positive constants, find the general solution of

$$y'' + k^2 y = \sin bx.$$

13. Use Principle of Superposition to find the general solution of

    a) $y'' + 4y = 4\cos 2x + 6\cos x + 8x^2 - 4x$.

    b) $y'' + 9y = 2\sin 3x + 4\sin x - 26e^{-2x} + 27x^3$.

***Answers to Exercises Set B***

1. $y = c_1 e^{2x} + c_2 e^{-5x} + \frac{1}{3} e^{4x}$

2. $y = c_1 \sin 2x + c_2 \cos 2x + \sin x$

3. $y = c_1 e^{-5x} + c_2 x e^{-5x} + 7x^2 e^{-5x}$;

4. $y = e^x(c_1 \cos 2x + c_2 \sin 2x) + 2 + 4x + 5x^2$

5. $y = c_1 e^{3x} + c_2 e^{-2x} - 4xe^{-2x}$

6. $y = c_1 e^x + c_2 e^{2x} + 2\sin 2x + 3\cos 2x$

7. $y = c_1 \sin x + c_2 \cos x + x \sin x$

8. $y = c_1 + c_2 e^{2x} + 2x - 3x^2$

9. $y = c_1 e^x + c_2 x e^x + 3x^2 e^x$

10. $y = e^x(c_1 \cos x + c_2 \sin x) - \frac{1}{2} x e^x \cos x$

11. $y = c_1 + c_2 e^{-x} + 2x^5 - 10x^4 + 40x^3 - 120x^2 + 242x.$

12. $y = \begin{cases} c_1 \sin kx + c_2 \cos kx + \frac{\sin bx}{k^2 - b^2} & \text{if } b \neq k \\ c_1 \sin kx + c_2 \cos kx - \frac{x \cos kx}{2k} & \text{if } b = k \end{cases}$

13. (a) $y = c_1 \sin 2x + c_2 \cos 2x + x \sin 2x + 2\cos x - 1 - x + 2x^2.$

    (b) $y = c_1 \sin 3x + c_2 \cos 3x - \frac{1}{3} x \cos 3x + \frac{1}{2} \sin x - 2e^{-2x} + 3x^3 - 2x.$

Chapter 15

# Method of Variation of Parameters

The method discussed in the last chapter is simple and has important engineering applications. But it applies only to constant-coefficient equations with special right sides$g(t)$. In this chapter we discuss the so-called **method of variation of parameters,** which is completely general (but more complicated). That is, it also applies to differential equations

$$y'' + p(t)y' + q(t)y = g(t) \tag{15.1}$$

with ***arbitrary variable*** functions$p$, $q$, and $g$ that are continuous on some interval $I$. The method gives a particular solution $y_p$ of

(15.1) on $I$ in the form

$$y_p = -y_1 \int \frac{y_2 g}{W} dt + y_2 \int \frac{y_1 g}{W} dt \tag{15.2}$$

where $y_1$, $y_2$ form a fundamental set of solutions of the homogenous equation

$$y'' + p(t)y' + q(t)y = 0 \tag{15.3}$$

corresponding to (15.1) and

$$W = W\left(y_1,\ y_2\right) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1' \tag{15.4}$$

is the **Wronskian** of $y_1$, $y_2$.

**Theorem** If the functions$p$, $q$, and $g$ are continuous on an open interval $I$, and if the functions $y_1$ and $y_2$ are a fundamental set of solutions of the homogeneous equation

$$y'' + p(t)y' + q(t)y = 0 \tag{15.5}$$

corresponding to the nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t) \tag{15.6}$$

then a particular solution of Eq. (15.6) is

$$y_p(t) = -y_1(t) \int_{t_0}^{t} \frac{y_2(s)\, g(s)}{w(y_1,\, y_2)(s)}\, ds + y_2(t) \int_{t_0}^{t} \frac{y_1(s)\, g(s)}{W(y_1,\, y_2)(s)}\, ds,$$

where $t_0$ is any conveniently chosen point in $I$. The general solution is

$$y = c_1 y_1(t) + c_2 y_2(t) + y_p(t).$$

**Attention!** Before applying (15.2), make sure that your equation is written in the standard form (15.1). If the equation is

$$F(t)\, y'' + G(t) y' + R(t) = S(t),$$

with $F(t) \neq 0$, divide by $F(t)$ and bring it to the form (15.1).

The integration in (15.2) may often cause difficulties. If you have a choice , apply the method of undetermined coefficients. It is simpler.

***Example* 1** (*Method of variation of parameters*)

Solve the differential equation

$$y'' + y = \sec t.$$

*Solution*

**Step 1:** The corresponding homogeneous equation is given by $y'' + y = 0$ and the corresponding characteristic equation is

$$\lambda^2 + 1 = 0$$

which gives

$$\lambda = \pm i.$$

Hence

$$y_h = e^{0t}(A\cos t + B\sin t) = A\cos t + B\sin t$$

**Step 2:** From Step 1, the fundamental set of solutions of the homogenous equation on any interval is

$$y_1 = \cos t, y_2 = \sin t.$$

Hence, we have the Wronskian

$$W(y_1\ , y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \cos t\cos t - \sin t\,(-\sin t) = 1.$$

Hence from (15.2), choosing the constants of integration to be zero, we have

$$\begin{aligned} y_p &= -\cos t\int \sin t\sec t\,dt + \sin t\int \cos t\ \sec t\,dt \\ &= -\cos t\int \frac{\sin t}{\cos t}\,dt + \sin t\int \frac{\cos t}{\cos t}\,dt \\ &= -\cos t(-\ln|\cos t|) + t\sin t \\ &= \cos t\ln|\cos t| + t\ \sin t \end{aligned}$$

**Step 3:** The general solution of the given nonhomogeneous equation is

$$y = y_h + y_p = [c_1 + \ln|\cos t|]\cos t + (c_2 + t)\sin t.$$

***Example* 2** Solve by the method of variation of parameters:

$$(D^2 + 4D + 4)y = \frac{e^{-2t}}{t^2}$$

*Solution*

**Step 1:** The solution of the corresponding homogeneous equation $(D^2 + 4D + 4)y = 0$ is

$$y_h = (c_1 + c_2 t)e^{-2t}.$$

**Step 2:** As the basis of solutions of the homogenous equation on any interval is

$$y_1 = e^{-2t}, y_2 = te^{-2t},$$

we have the Wronskian

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t}(1-2t) \end{vmatrix} = e^{-4t}.$$

Hence from (15.2), choosing the constants of integration to be zero, we have

$$y_p = -e^{-2t} \int \frac{te^{-2t}}{e^{-4t}} \frac{e^{-2t}}{t^2} dt + te^{-2t} \int \frac{e^{-2t}}{e^{-4t}} \frac{e^{-2t}}{t^2} dt$$

$$= -e^{-2t} \int \frac{1}{t} dt + te^{-2t} \int \frac{1}{t^2} dt = -e^{-2t} \ln|t| + te^{-2t}(-t^{-1})$$

$$= (-\ln|t| - 1)\, e^{-2t}$$

**Step 3**: The general solution of the given nonhomogeneous equation is

$$y = y_h + y_p = (c_1 + c_2 t - \ln|t| - 1)e^{-2t}.$$

**Euler-Cauchy Equations**

Now we consider the differential equation in which the homogeneous equation is Euler-Cauchy.

***Example* 3** Solve the following nonhomogeneous Euler-Cauchy equation

$$t^2 y'' - 4ty' + 6y = 21t^{-4}$$

by the method of variation of parameters.

*Solution*

**Step 1:** The solution of the corresponding homogeneous Euler-Cauchy equation $t^2 y'' - 4ty' + 6y = 0$ is

$$y_h = c_1 t^3 + c_2 t^2.$$

**Step 2:** As the basis of solutions of the homogenous equation on any interval is

$$y_1 = t^3, y_2 = t^2,$$

we have the Wronskian

$$W(y_1\ , y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t^3 & t^2 \\ 3t^2 & 2t \end{vmatrix} = -t^4.$$

Now to use (15.2), we have to bring the given differential equation to the standard form (15.1) [1]

Dividing by $t^2$, we obtain

$$y'' - \frac{4}{t}y' + \frac{6}{t^2}y = 21t^{-6}.$$

Comparing with the standard equation (15.1), we have

$$g(t) = 21\,t^{-6}.$$

Hence from (15.2), choosing the constants of integration to be zero, we have

$$\begin{aligned} y_p &= -t^3 \int \frac{t^2 21t^{-6}}{-t^4} dt + t^2 \int t^3 \frac{21t^{-6}}{-t^4}\, dt \\ &= 21t^3 \int t^{-8} dt - 21t^2 \int t^7\, dt \\ &= 21t^3 \frac{t^{-7}}{-7} - 21t^2 \frac{t^6}{-6} \\ &= \frac{1}{2}t^{-4} \end{aligned}$$

**Step 3** The general solution of the given nonhomogeneous equation is

$$y = y_h + y_p = c_1 t^3 + c_2 t^2 + \frac{1}{2}t^{-4}.$$

[1] **Attention!** Homogeneous equation corresponding the given differential equation is not readily in the standard form of Euler-Cauchy equation.

***Example* 4** Solve

$$ty'' - y' = (3+t)t^2e^t.$$

*Solution*

When multiplied by $t$, the given equation becomes the Euler-Cauchy equation

$$t^2y'' - ty' = (3+t)t^3e^t. \tag{15.7}$$

**Step 1:** The solution of the corresponding homogeneous Euler-Cauchy equation $t^2y'' - ty' = 0$ is

$$y_h = c_1 + c_2t^2.$$

**Step 2:** As the basis of solutions of the homogenous equation on any interval is

$$y_1 = 1, y_2 = t^2,$$

we have the Wronskian

$$W(y_1\ , y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} 1 & t^2 \\ 0 & 2t \end{vmatrix} = 2t.$$

Now to use (15.2), we have to bring the given differential equation to the standard form (15.1). Dividing (15.7) by $t^2$, we obtain

$$y'' - \frac{1}{t}y' = (3+t)te^t.$$

Comparing with the standard equation (15.1), we have

$$g(t) = (3+t)te^t.$$

Hence from (15.2), choosing the constants of integration to be zero, we have

$$y_p = -\int \frac{t^2(3+t)te^t}{2t}dt + t^2 \int \frac{(3+t)te^t}{2t}\,dt$$

$$= \frac{1}{2}\int (3+t)t^2e^t + \frac{t^2}{2}\int (3+t)e^t dt.$$

The rest of the work is left as an exercise to the student.

**Exercises**

In Exercises 1-10, find the general solution of the nonhomogeneous equation using method of variation of parameters.

1. $y'' - 4y' + 4y = \frac{e^{2t}}{t}$
2. $y'' + 2y' + y = e^{-t}\cos t$
3. $y'' - 2y' + y = \frac{e^t}{t^3}$
4. $(D^2 - 2D + 1)y = 3t^{3/2}e^t$
5. $(D^2 + 4D + 4)y = 2\frac{e^{-2t}}{t^2}$
6. $y'' + 9y = \sec 3t$
7. $y'' + 9y = \csc 3t$
8. $y'' - 4y' + 5y = e^{2t}\csc t$

9. $(D^2 + 6D + 9)y = 16\frac{e^{-3t}}{t^2+1}$

10. $(D^2 + 2D + 2)y = 4e^{-t}\sec^3 t$

In Exercises 11-15, find the general solution of the nonhomogeneous Euler-Cauchy equation using method of variation of parameters.

11. $4t^2y'' + 8ty' - 3y = 7t^2 - 15t^3$

12. $(t^2D^2 - 2tD + 2)y = t^3\cos t$

13. $(t^2D^2 + tD - 9)y = 48t^5$

14. $(t^2D^2 - 4tD + 6)y = 7t^4\sin t$

15. $(t^2D^2 + tD - 1)y = \frac{1}{t^2}$

In each of Exercises 16-18, verify that the given functions $y_1$ and $y_2$ satisfy the corresponding homogeneous equation; then find a particular solution of the given non homogeneous equation (In Exercise 18, $g$ is an arbitrary continuous function.)

16. $t^2y'' - t(t+2)y' + (t+2)y = 2t^3, t > 0$; $y_1(t) = t,\;\; y_2(t) = te^t$

17. $x^2y'' + xy' + (x^2 - 0.25)y = 3x^{3/2}\sin x, x > 0$; $y_1(x) = x^{-1/2}\sin x,\;\; y_2(x) = x^{-1/2}\cos x$

18. $(1-x)y'' + xy' - y = g(x),\;\; 0 < x < 1; y_1(x) = e^x,\;\; y_2(x) = x$.

***Answers***

1. $(c_1 + c_2 t + t \ln|t| - t)\, e^{2t}$

2. $(c_1 + c_2 t - \cos t)\, e^{-t}$

3. $(c_1 + c_2 t)\, e^t + \frac{1}{2}\frac{e^t}{t}$

4. $\left(c_1 + c_2 t + \frac{12}{35} t^{7/2}\right) e^t$

5. $(c_1 + c_2 t - 2\ln|t|)\, e^{-2t}$

11. $c_1 t^{1/2} + c_2 t^{-3/2} + \frac{1}{3}(t^2 - t^3)$

12. $c_1 t + c_2 t^2 - t\cos t$

13. $c_1 t^3 + c_2 t^{-3} + 3t^5$

16. $y_p(t) = -2t^2$

17. $y_p(x) = -\frac{3}{2} x^{1/2} \cos x$

18. $y_p(x) = \int \frac{xe^t - te^x}{(1-t)^2\, e^t}\, g(t)\, dt$

# Chapter 16

# Series Solutions Near an Ordinary Point

## 16.1 Series Solutions Near an Ordinary Point, Part I

In previous chapters we described methods of solving second-order linear differential equations with constant coefficients. We now consider methods of solving second-order linear equations when the coefficients are functions of the independent variable. In this chapter we will denote the independent variable by $x$. It is sufficient to consider the homogeneous equation

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0, \tag{16.1}$$

since the procedure for the corresponding nonhomogeneous equation is similar.

Many problems in mathematical physics lead to equations of the form (16.1) having polynomial coefficients; examples include the Bessel equation

$$x^2y'' + xy' + (x^2 - \upsilon^2)y = 0, \tag{16.2}$$

where $\upsilon$ is a constant, and the Legendre equation

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0, \tag{16.3}$$

where $\alpha$ is a constant. For this reason, as well as to simplify the algebraic computations, we primarily consider the case in which the functions $P$, $Q$, and $R$ are polynomials. However, as we will see, the method of solution is also applicable when $P$, $Q$, and $R$ are general analytic functions.

For the present, then, suppose that $P$, $Q$, and $R$ are polynomials and that there is no factor $(x - c)$ that is common to all three of them. If there is such a common factor $(x - c)$, then divide it out before proceeding. Suppose also that we wish to solve equation (16.1) in the neighborhood of a point $x_0$. The solution of equation (16.1) in an interval containing $x_0$ is closely associated with the behavior of $P$ in that interval.

A point $x_0$ such that $P(x_0) \neq 0$ is called an **ordinary point.** Since $P$ is continuous, it follows that there is an open interval containing $x_0$ in which $P(x)$ is never zero. In that interval, which we will call $I$, we can divide equation (16.1) by $P(x)$ to obtain

$$y'' + p(x)y' + q(x)y = 0, \tag{16.4}$$

where $p(x) = Q(x)/P(x)$ and $q(x) = R(x)/P(x)$ are continuous functions on $I$. Hence, according to the *existence and uniqueness theorem* (Theorem 1 in Page 161), there exists a unique solution of equation (16.1) in the interval $I$ that also satisfies the initial conditions $y(x_0) = y_0$ and $y'(x_0) = y_0'$ for arbitrary values of $y_0$ and $y_0'$. We discuss the solution of equation (16.1) in the neighborhood of an ordinary point. On the other hand, if P( x0) = 0, then x0 is called a singular point of equation (16.1). In this case, because $(x - x_0)$ is not a factor of $P$, $Q$, and $R$, at least one of $Q(x_0)$ and $R(x_0)$ is not zero. Consequently, at least one of the coefficients $p$ and $q$ in equation (16.4) becomes unbounded as $x \to x_0$, and therefore *existence and uniqueness theorem* does not apply in this case. The method to find solutions of equation (16.1) in the neighborhood of a singular point is not discussed in this study material.

We now take up the problem of solving equation (16.1) in the neighborhood of an ordinary point $x_0$. We look for solutions of the form

$$y = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)^n + \cdots = \sum_{n=0}^{\infty} a_n(x - x_0)^n \tag{16.5}$$

and assume that the series converges in the interval $|x - x_0| < \rho$ for some $\rho > 0$.

While at first sight it may appear unattractive to seek a solution in the form of a power series, this is actually a convenient and useful form for a solution. Within their intervals of convergence, power series behave very much like polynomials and are easy to manipulate both analytically and numerically. Indeed, even if we can obtain a solution in terms of elementary functions, such as exponential or trigonometric functions, we are likely to need a power series or some equivalent expression if we want to evaluate the solution numerically or to plot its graph.

The most practical way to determine the coefficients $a_n$ is to substitute the series (16.5) and its derivatives for $y$, $y'$, and $y''$ in equation (16.1). The following examples illustrate this process. The operations, such as differentiation, that are involved in the procedure are justified so long as we stay within the interval of convergence. The differential equations in these examples are also of considerable importance in their own right.

***Example* 1** Find a series solution of the equation

$$y'' + y = 0, \quad -\infty < x < \infty. \tag{16.6}$$

*Solution*

We look for a solution of the form of a power series about

$x_0 = 0$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots = \sum_{n=0}^{\infty} a_n x^n \quad (16.7)$$

and assume that the series converges in some interval $|x| < \rho$. Differentiating equation (16.7) term by term, we obtain

$$y' = \frac{dy}{dx} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1} + \cdots = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad (16.8)$$

and

$$y'' = 2a_2 + 3 \cdot 2 a_3 x + \cdots + n(n-1) a_n x^{n-2} + \cdots = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}. \quad (16.9)$$

Substituting the series (16.7), (16.8), and (16.9) into the equation (16.6) and obtain

$$(2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x^2 + \cdots) + (a_0 + a_1 x + a_2 x^2 + \cdots) = 0.$$

That is,

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0.$$

To combine the two series, we need to rewrite at least one of them so that both series display the same generic term. Thus, in the first sum, we shift the index of summation by replacing $n$ by $n+2$

and starting the sum at 0 rather than 2. We obtain

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}a_n x^n = 0$$

or

$$\sum_{n=0}^{\infty}\left((n+2)(n+1)a_{n+2} + a_n\right)x^n = 0.$$

For this equation to be satisfied for all $x$, the coefficient of each power of $x$ must be zero; hence we conclude that

$$(n+2)(n+1)a_{n+2} + a_n = 0, \quad n = 0,\ 1,\ 2,\ 3,\ \ldots \tag{16.10}$$

Equation (16.10) is referred to as a **recurrence relation.** The successive coefficients can be evaluated one by one by writing the recurrence relation first for $n = 0$, then for $n = 1$, and so forth. In this example equation (16.10) relates each coefficient to the second one before it. Thus the even numbered coefficients ($a_0$, $a_2$, $a_4$, ...) and the odd-numbered ones ($a_1$, $a_3$, $a_5$, ...) are determined separately. For the even-numbered coefficients we have

$$a_2 = \frac{a_0}{2\cdot 1} = \frac{a_0}{2!},\quad a_4 = -\frac{a_2}{4\cdot 3} = \frac{a_0}{4!},\quad a_6 = -\frac{a_4}{6\cdot 5} = -\frac{a_0}{6!},\quad \ldots$$

These results suggest that in general, if $n = 2k$, then

$$a_n = a_{2k} = \frac{(-1)^k}{(2k)!}a_0,\quad k = 1,\ 2,\ 3,\ \ldots \tag{16.11}$$

We can prove equation (16.11) by mathematical induction.

First, observe that it is true for $k = 1$. Next, assume that it is true for an arbitrary value of $k$ and consider the case $k+1$. We have

$$a_{2k+2} = -\frac{a_{2k}}{(2k+2)(2k+1)} = -\frac{(-1)^k}{(2k+2)(2k+1)(2k)!}a_0 = \frac{(-1)^k}{(2k+2)!}a_0.$$

Hence equation (16.11) is also true for $k+1$, and consequently it is true for all positive integers $k$.

Similarly, for the odd-numbered coefficients

$$a_3 = -\frac{a_1}{2 \cdot 3} = -\frac{a_1}{3!}, \quad a_5 = -\frac{a_3}{5 \cdot 4} = \frac{a_1}{5!}, \quad a_7 = -\frac{a_5}{7 \cdot 6} = -\frac{a_1}{7!}, \quad \ldots$$

and in general, if $n = 2k+1$, then

$$a_n = a_{2k+1} = \frac{(-1)^k}{(2k+1)!}a_1, \quad k = 1,\ 2,\ 3,\ \ldots$$

Substituting these coefficients into equation (16.7), we have

$$\begin{aligned} y &= a_0 + a_1 x - \frac{a_0}{2!}x^2 - \frac{a_1}{3!}x^3 + \frac{a_0}{4!}x^4 + \frac{a_1}{5!}x^5 \\ &\quad + \cdots + \frac{(-1)^n a_0}{(2n)!}x^{2n} + \frac{(-1)^n a_1}{(2n)!}x^{2n+1} + \cdots \end{aligned}$$

$$
\begin{aligned}
&= a_0\left[1-\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots+\frac{(-1)^n}{(2n)!}x^{2n}+\cdots\right] \\
&\quad +a_1\left[x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots+\frac{(-1)^n}{(2n+1)!}x^{2n+1}+\cdots\right] \\
&= a_0\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}+a_1\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}. \qquad (16.12)
\end{aligned}
$$

We identify two series solutions of equation (16.6):

$$y_1(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} \quad \text{and} \quad y_2(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}.$$

**Remark** In Example 1 we knew from the start that $\sin x$ and $\cos x$ form a fundamental set of solutions of equation (16.6). However, if we had not known this and had simply solved equation (16.6) using series methods, we would still have obtained the solution (16.12). In recognition of the fact that the differential equation (16.6) often occurs in applications, we might decide to give the two solutions of equation (16.12) special names, perhaps

$$C(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}, \quad S(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}. \qquad (16.13)$$

Then we might ask what properties these functions have. For instance, can we be sure that $C(x)$ and $S(x)$ form a fundamental set of solutions? It follows at once from the series expansions that $C(0)=1$ and $S(0)=0$. By differentiating the series for $C(x)$ and

$S(x)$ term by term, we find that

$$S'(x) = C(x), \;\; C'(x) = -S(x). \tag{16.14}$$

Thus at $x = 0$, we have $S'(0) = C(0) = 1$ and $C'(0) = -S(0) = 0$. Consequently, the Wronskian of $C$ and $S$ at $x = 0$ is

$$W[C, \; S](0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1, \tag{16.15}$$

so these functions do indeed form a fundamental set of solutions. By substituting $-x$ for $x$ in each of equations (16.15), we obtain $C(-x) = C(x)$ and $S(-x) = -S(x)$. Moreover, by calculating with the infinite series, we can show that the functions $C(x)$ and $S(x)$ have all the usual analytical and algebraic properties of the cosine and sine functions, respectively.

**Definition of $\sin x$ and $\cos x$ in terms of initial value problem** Although you probably first saw the sine and cosine functions defined in a more elementary manner in terms of right triangles, it is interesting that these functions can be defined as solutions of a certain simple second-order linear differential equation. To be precise, the function $\sin x$ can be defined as the unique solution of the initial-value problem $y" + y = 0, \; y(0) = 0, \; y'(0) = 1$; similarly, $\cos x$ can be defined as the unique solution of the initial-value problem $y'' + y = 0, \; y(0) = 1, \; y'(0) = 0$. Many other functions that are important in mathematical physics are also defined as solutions of certain initial-value problems. For most of these functions there is no simpler or more elementary way to approach them.

***Example* 2** Find a series solution in powers of $x$ of Airy's[1] equation

$$y'' - xy = 0, \quad -\infty < x < \infty. \tag{16.16}$$

*Solution*

For this equation $P(x) = 1,\ Q(x) = 0$, and $R(x) = -x$; hence every point is an ordinary point. We assume that

$$y = \sum_{n=0}^{\infty} a_n x^n \tag{16.17}$$

and that the series converges in some interval $|x| < \rho$. The series for $y$" is given by equation (16.9); as explained in the preceding example, we can rewrite it as

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n. \tag{16.18}$$

Substituting the series (16.17) and (16.18) for $y$ and $y''$ into the

[1] Sir George Biddell Airy (1801 -1892), an English astronomer and mathematician, was director of the Greenwich Observatory from 1835 to 1881. He studied the equation named for him in an 1838 paper on optics. One reason why Airy's equation is of interest is that for $x$ negative the solutions are similar to trigonometric functions, and for $x$ positive they are similar to hyperbolic functions.

left-hand side of equation (16.16), we obtain

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - x\sum_{n=0}^{\infty}a_n x^n$$
$$= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty}a_n x^{n+1} \quad (16.19)$$

Next, we shift the index of summation in the second series on the right-hand side of equation (16.19) by replacing $n$ by $n-1$ and starting the summation at 1 rather than zero. Thus we write equation (16.16) as

$$2\cdot 1a_2 + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty}a_{n-1}x^n = 0.$$

Again, for this equation to be satisfied for all $x$ in some interval, the coefficients of like powers of $x$ must be zero; hence $a_2 = 0$, and we obtain the recurrence relation

$$(n+2)(n+1)a_{n+2} - a_{n-1} = 0 \quad \text{for} \quad n = 1,\ 2,\ 3,\ \ldots \quad (16.20)$$

Since $a_{n+2}$ is given in terms of $a_{n-1}$, the $a'$s are determined in steps of three. Thus $a_0$ determines $a_3$, which in turn determines $a_6, \ldots$; $a_1$ determines $a_4$, which in turn determines $a_7, \ldots,$; and $a_2$ determines $a_5$, which in turn determines $a_8, \ldots$. Since $a_2 = 0$, we immediately conclude that $a_5 = a_8 = a_{11} = \cdots = 0$.

For the sequence $a_0, a_3, a_6, a_9, \ldots$ we set $n = 1, 4, 7, 10, \ldots$

in the recurrence relation:

$$a_3 = \frac{a_0}{2\cdot 3}, \quad a_6 = \frac{a_3}{5\cdot 6} = \frac{a_0}{2\cdot 3\cdot 5\cdot 6},$$

$$a_9 = \frac{a_6}{8\cdot 9} = \frac{a_0}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}, \quad \ldots$$

These results suggest the general formula

$$a_{3n} = \frac{a_0}{2\cdot 3\cdot 5\cdot 6\cdots(3n-1)(3n)}, \quad n \geq 4.$$

For the sequence $a_1,\ a_4,\ a_7,\ a_{10},\ \ldots$, we set $n = 2,\ 5,\ 8,\ 11,\ \ldots$ in the recurrence relation:

$$a_4 = \frac{a_1}{3\cdot 4}, \quad a_7 = \frac{a_4}{6\cdot 7} = \frac{a_1}{3\cdot 4\cdot 6\cdot 7},$$

$$a_{10} = \frac{a_7}{9\cdot 10} = \frac{a_1}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10}, \quad \ldots$$

In general, we have

$$a_{3n+1} = \frac{a_1}{3\cdot 4\cdot 6\cdot 7\cdots(3n)(3n+1)}, \quad n \geq 4.$$

Thus the general solution of Airy's equation is

$$y(x) =$$

$$a_0\left[1 + \frac{x^3}{2\cdot 3} + \frac{x^6}{2\cdot 3\cdot 5\cdot 6} + \cdots + \frac{x^{3n}}{2\cdot 3\cdot 5\cdot 6\cdots(3n-1)(3n)} + \cdots\right]$$

$$+a_1\left[x + \frac{x^4}{3\cdot 4} + \frac{x^7}{3\cdot 4\cdot 6\cdot 7} + \cdots + \frac{x^{3n+1}}{3\cdot 4\cdot 6\cdot 7\cdots(3n)(3n+1)} + \cdots\right]$$

$$= \quad a_0 y_1(x) + a_1 y_2(x) \tag{16.21}$$

where $y_1(x)$ and $y_2(x)$ are the first and second bracketed expressions in equation (16.21).

Having obtained these two series solutions, we can now investigate their convergence. Because of the rapid growth of the denominators of the terms in the series for $y_1(x)$ and for $y_2(x)$, we might expect these series to have a large radius of convergence. Indeed, it is easy to use the ratio test[2] to show that both of these series converge for all $x$.

Assume for the moment that the series for $y_1$ and $y_2$ do converge for all $x$. Then, by choosing first $a_0 = 1$, $a_1 = 0$ and then $a_0 = 0$, $a_1 = 1$, it follows that $y_1$ and $y_2$ are individually solutions of equation (16.16). Notice that $y_1$ satisfies the initial conditions $y_1(0) = 1$, $y_1'(0) = 0$ and that $y_2$ satisfies the initial conditions $y_2(0) = 0$, $y_2'(0) = 1$. Thus $W[y_1, y_2](0) = 1 \neq 0$, and consequently $y_1$ and $y_2$ are a fundamental set of solutions. Hence the general solution of Airy's equation is

$$y = a_0 y_1(x) + a_1 y_2(x) \quad -\infty < x < \infty.$$

[2]If the sequence $\left|\frac{a_{n+1}}{a_n}\right|$ , $n = 1,\ 2,\ \ldots$ is convergent with the limit $L$, then the radius of convergence $R$ of the power series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \ \ldots$$

is $R = \frac{1}{L}$ when $L > 0$ and $R = \infty$ when $L = 0$.

***Example* 3** Find a solution of Airy's equation in powers of $x - 1$.

*Solution*

The point $x = 1$ is an ordinary point of equation (16.16), and thus we look for a solution of the form

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n,$$

where we assume that the series converges in some interval $|x-1| < \rho$. Then

$$y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-1)^n,$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n$$

Substituting for $y$ and $y''$ in equation (16.16), we obtain

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n = x \sum_{n=0}^{\infty} a_n (x-1)^n. \qquad (16.22)$$

Now to equate the coefficients of like powers of $(x - 1)$, we must express $x$, the coefficient of $y$ in equation (16.16), in powers of $x - 1$; that is, we write $x = 1 + (x - 1)$. Note that this is precisely the Taylor series for $x$ about $x = 1$. Then equation (16.22) takes the form

$$\begin{aligned}\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n &= (1+(x-1))\sum_{n=0}^{\infty}a_n(x-1)^n\\ &= \sum_{n=0}^{\infty}a_n(x-1)^n+\sum_{n=0}^{\infty}a_n(x-1)^{n+1}.\end{aligned}$$

Shifting the index of summation in the second series on the right gives

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n=\sum_{n=0}^{\infty}a_n(x-1)^n+\sum_{n=1}^{\infty}a_{n-1}(x-1)^n.$$

Equating coefficients of like powers of $x-1$, we obtain

$$\begin{aligned}2a_2 &= a_0,\\ (3\cdot 2)a_3 &= a_1+a_0,\\ (4\cdot 3)a_4 &= a_2+a_1,\\ (5\cdot 4)a_5 &= a_3+a_2,\\ &\vdots\end{aligned}$$

The general recurrence relation is

$$(n+2)(n+1)a_{n+2}=a_n+a_{n-1}\ \text{ for }\ n\geq 1\,. \tag{16.23}$$

Solving for the first few coefficients $a_n$ in terms of $a_0$ and $a_1$, we find that

$$a_2 = \frac{a_0}{2}, \quad a_3 = \frac{a_1}{6} + \frac{a_0}{6}, \qquad a_4 = \frac{a_2}{12} + \frac{a_1}{12} = \frac{a_0}{24} + \frac{a_1}{12},$$

$$a_5 = \frac{a_3}{20} + \frac{a_2}{20} = \frac{a_0}{30} + \frac{a_1}{120}.$$

Hence

$$\begin{aligned} y &= a_0 \left[ 1 + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{24} + \frac{(x-1)^5}{30} + \cdots \right] \\ &\quad + a_1 \left[ (x-1) + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{12} + \frac{(x-1)^5}{120} + \cdots \right] \end{aligned} \tag{16.24}$$

In general, when the recurrence relation has more than two terms, as in equation (16.23), the determination of a formula for an in terms $a_0$ and a1 will be fairly complicated, if not impossible. In this example such a formula is not readily apparent. Lacking such a formula, we cannot test the two series in equation (16.24) for convergence by direct methods such as the ratio test. However, we shall see shortly that even without knowing the formula for $a_n$, it is possible to establish that the two series in equation (16.24) converge for all $x$. Further, they define functions $y_3$ and $y_4$ that are a fundamental set of solutions of the Airy equation (16.16). Thus

$$y = a_0 y_3(x) + a_1 y_4(x)$$

is the general solution of Airy's equation for $\infty < x < \infty$.

## 16.2 Series Solutions Near an Ordinary Point, Part II

In the preceding section we considered the problem of finding solutions of

$$P(x)y'' + Q(x)y' + R(x)y = 0, \tag{16.25}$$

where $P$, $Q$, and $R$ are polynomials, in the neighborhood of an ordinary point $x_0$. Assuming that equation (16.25) does have a solution $y = \phi(x)$ and that $\phi$ has a Taylor series

$$\phi(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{16.26}$$

that converges for $|x - x_0| < \rho$, where $\rho > 0$, we found that the an can be determined by directly substituting the series (16.26) for $y$ in equation (16.25).

Let us now consider how we might justify the statement that if $x_0$ is an ordinary point of equation (16.25), then there exist solutions of the form (16.26). We also consider the question of the radius of convergence of such a series. In doing this, we are led to a generalization of the definition of an ordinary point.

Suppose, then, that there is a solution of equation (16.25) of the form (16.26). By differentiating equation (16.26) $m$ times and setting $x$ equal to $x_0$, we obtain

$$m! a_m = \phi^{(m)}(x_0). \tag{16.27}$$

Hence, to compute $a_n$ in the series (16.26), we must show that we can determine $\phi^{(n)}(x_0)$ for $n = 0,\ 1,\ 2,\ \ldots$ from the differential equation (16.25).

Suppose that $y = \phi(x)$ is a solution of equation (16.25) satisfying the initial conditions $y(x_0) = y_0,\ y'(x_0) = y'_0$. Then $a_0 = y_0$ and $a_1 = y'_0$. If we are solely interested in finding a solution of equation (16.25) without specifying any initial conditions, then $a_0$ and $a_1$ remain arbitrary. To determine $\phi^{(n)}(x_0)$ and the corresponding $a_n$ for $n = 2,\ 3,\ \ldots,$ we turn to equation (16.25) with the goal of finding a formula for $\phi''(x),\ \phi'''(x),\ \ldots$ .

Since $\phi$ is a solution of equation (16.25), we have

$$P(x)\phi''(x) + Q(x)\phi'(x) + R(x)\phi(x) = 0.$$

For the interval about $x_0$ for which $P$ is nonzero, we can write this equation in the form

$$\phi''(x) = -p(x)\phi'(x) - q(x)\phi(x). \tag{16.28}$$

where $p(x) = Q(x)/P(x)$ and $q(x) = R(x)/P(x)$ Observe that, at $x = x_0$, the right-hand side of equation (16.28) is known, thus allowing us to compute $\phi''(x_0)$ : Setting $x$ equal to $x_0$ in equation (16.28) gives

$$\phi''(x_0) = -p(x_0)\phi'(x_0) - q(x_0)\phi(x_0) = -p(x_0)a_1 - q(x_0)a_0.$$

Hence, using equation (16.27) with $m = 2$, we find that $a_2$ is given by

$$2!a_2 = \phi''(x_0) = -p(x_0)a_1 - q(x_0)a_0. \tag{16.29}$$

To determine $a_3$, we differentiate equation (16.28) and then set $x$ equal to $x_0$, obtaining

$$\begin{aligned} 3!a_3 &= \phi'''(x_0) = -\left(p(x)\phi'(x) + q(x)\phi(x)\right)'\Big|_{x=x_0} \\ &= -2!p(x_0)a_2 - \left(p'(x_0) + q(x_0)\right)a_1 - q'(x_0)a_0. \end{aligned} \tag{16.30}$$

Substituting for $a_2$ from equation (16.29) gives $a_3$ in terms of $a_1$ and $a_0$.

Since $P$, $Q$, and $R$ are polynomials and $P(x_0) \neq 0$, all the derivatives of $p$ and $q$ exist at $x_0$. Hence we can continue to differentiate equation (16.28) indefinitely, determining after each differentiation the successive coefficients $a_4,\ a_5,\ \ldots$ by setting $x$ equal to $x_0$.

***Example* 4** Let $y = \phi(x)$ be a solution of the initial value problem

$$(1 + x^2)y'' + 2xy' + 4x^2y = 0,\ \ y(0) = 0,\ \ y'(0) = 1.$$

Determine $\phi''(0),\ \ \phi'''(0)$, and $\phi^{(4)}(0)$.

*Solution*

To find $\phi''(0)$, simply evaluate the differential equation when $x = 0:$

$$(1 + 0^2)\phi''(0) + 2 \cdot 0\phi'(0) + 4 \cdot 0^2\phi(0) = 0,$$

so

$$\phi''(0) = 0.$$

To find $\phi'''(0)$, differentiate the differential equation with respect to $x$ :

$$(1+x^2)\,\phi'''(x)+2x\phi''(x)+2x\phi''(x)+2\phi'(x)+4x^2\phi'(x)+8\phi(x) = 0. \tag{16.31}$$

Then evaluate the resulting equation (16.31) at $x = 0$ :

$$\phi'''(0) + 2\phi'(0) = 0.$$

Thus

$$\phi'''(0) = -2\phi'(0) = -2$$

because $\phi'(0) = y'(0) = 1$.

Finally, to find $\phi^{(4)}(0)$, first differentiate equation (16.31) with respect to $x$ :

$$\begin{aligned}(1+x^2)\,\phi^{(4)}(x) \quad & +2x\phi'''(x) + 4x\phi'''(x) + 4\phi''(x) \\ & +(2+4x^2)\phi''(x) + 8x\phi'(x) + 8x\phi'(x) + 8\phi(x) = 0.\end{aligned}$$

Evaluating this equation at $x = 0$ we find

$$\phi^{(4)}(0) + 6\phi''(0) + 8\phi(0) = 0.$$

Finally, using $\phi(0) = 0$ and $\phi''(0) = 0$, we conclude that $\phi^{(4)}(0) =$

0.

Notice that the important property that we used in determining the $a_n$ was that we could compute infinitely many derivatives of the functions $p$ and $q$. It might seem reasonable to relax our assumption that the functions $p$ and $q$ are ratios of polynomials and simply require that they be infinitely differentiable in the neighborhood of $x_0$. Unfortunately, this condition is too weak to ensure that we can prove the convergence of the resulting series expansion for $y = \phi(x)$. What is needed is to assume that the functions $p$ and $q$ are *analytic* at $x_0$; that is, they have Taylor series expansions that converge to them in some interval about the point $x_0$ :

$$p(x) = p_0 + p_1(x - x_0) + \cdots + p_n(x - x_0)^n + \cdots = \sum_{n=0}^{\infty} p_n(x - x_0)^n, \tag{16.32}$$

$$q(x) = q_0 + q_1(x - x_0) + \cdots + q_n(x - x_0)^n + \cdots = \sum_{n=0}^{\infty} q_n(x - x_0)^n. \tag{16.33}$$

With this idea in mind, we can generalize the definitions of an ordinary point and a singular point of equation (16.25) as follows: if the functions $p(x) = Q(x)/P(x)$ and $q(x) = R(x)/P(x)$ are analytic at $x_0$, then the point $x_0$ is said to be an **ordinary point** of the differential equation (16.25); otherwise, it is a **singular point.**

Now let us turn to the question of the interval of convergence of the series solution. One possibility is actually to compute the series solution for each problem and then to apply one of the tests for convergence of an infinite series to determine its radius of convergence. Unfortunately, these tests require us to obtain an expression for the general coefficient $a_n$ as a function of $n$, and this task is often quite difficult, if not impossible. However, the question can be answered at once for a wide class of problems by the following theorem.

**Theorem 1** If $x_0$ is an ordinary point of the differential equation (16.25)

$$P(x)y'' + Q(x)y' + R(x)y = 0,$$

that is, if $p(x) = Q(x)/P(x)$ and $q(x) = R(x)/P(x)$ are analytic at $x_0$, then the general solution of equation (16.25) is

$$y = \sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0y_1(x) + a_1y_2(x),$$

where $a_0$ and $a_1$ are arbitrary, and $y_1$ and $y_2$ are two power series solutions that are analytic at $x_0$. The solutions $y_1$ and $y_2$ form a fundamental set of solutions. Further, the radius of convergence for each of the series solutions $y_1$ and $y_2$ is at least as large as the minimum of the radii of convergence of the series for $p$ and $q$.

***Example* 5** What is the radius of convergence of the Taylor series for $(1 + x^2)^{-1}$ about $x = 0$?

*Solution*

One way to proceed is to find the Taylor series in question, namely,

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots$$

Then it can be verified by the ratio test that $\rho = 1$. Another approach is to note that the zeros of $1 + x^2$ are $x = \pm i$. Since the distance in the complex plane from 0 to $i$ or to $-i$ is 1, the radius of convergence of the power series about $x = 0$ is 1.

***Example* 6** What is the radius of convergence of the Taylor series for $(x^2 - 2x + 2)^{-1}$ about $x = 0$? about $x = 1$?

*Solution*

First notice that

$$x^2 - 2x + 2 = 0$$

has solutions $x = 1 \pm i$. The distance in the complex plane from $x = 0$ to either $x = 1 + i$ or $x = 1 - i$ is $\sqrt{2}$; hence the radius of convergence of the Taylor series expansion $\sum_{n=0}^{\infty} a_n x^n$ about $x = 0$ is $\sqrt{2}$.

The distance in the complex plane from $x = 1$ to either $x = 1+i$ or $x = 1 - i$ is 1; hence the radius of convergence of the Taylor series expansion $\sum_{n=0}^{\infty} b_n (1 - x)^n$ about $x = 1$ is 1.

***Example* 7** Determine a lower bound for the radius of convergence of series solutions about $x = 0$ for the Legendre equation

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0,$$

where $\alpha$ is a constant.

*Solution*

Note that $P(x) = 1 - x^2$, $Q(x) = -2x$, and $R(x) = \alpha(\alpha + 1)$ are polynomials, and that the zeros of $P$, namely, $x = \pm 1$, are a distance 1 from $x = 0$. Hence a series solution of the form $\sum_{n=0}^{\infty} a_n x^n$ converges at least for $|x| < 1$, and possibly for larger values of $x$. Indeed, it can be shown that if $\alpha$ is a positive integer, one of the series solutions terminates after a finite number of terms, that is, one solution is a polynomial, and hence converges not just for $|x| < 1$ but for all $x$. For example, if $\alpha = 1$, the polynomial solution is $y = x$.

***Example* 8** Determine a lower bound for the radius of convergence of series solutions of the differential equation

$$(1 + x^2)y'' + 2xy' + 4x^2 y = 0 \tag{16.34}$$

about the point $x = 0$; about the point $x = -\frac{1}{2}$.

*Solution*

Again $P$, $Q$, and $R$ are polynomials, and $P$ has zeros at $x = \pm i$. The distance in the complex plane from 0 to $\pm i$ is 1, and from $-\frac{1}{2}$ to $\pm i$ is $\sqrt{1 + \frac{1}{4}} = \frac{\sqrt{5}}{2}$. Hence in the first case the series $\sum_{n=0}^{\infty} a_n x^n$ converges at least for $|x| < 1$, and in the second case the series $\sum_{n=0}^{\infty} b_n \left(x + \frac{1}{2}\right)^n$ converges at least for $\left|x + \frac{1}{2}\right| < \frac{\sqrt{5}}{2}$.

An interesting observation that we can make about equation (16.34) is the following. Suppose that initial conditions $y(0) = y_0$ and $y'(0) = y_0'$ are given. Since $1 + x^2 \neq 0$ for all $x$, there exists a unique solution of the initial-value problem on $-\infty < x < \infty$. On the other hand, it only guarantees a series solution of the form $\sum_{n=0}^{\infty} a_n x^n$ (with $a_0 = y_0$, $a_1 = y'_0$) for $-1 < x < 1$. The unique solution on the interval $-\infty < x < \infty$ may not have a power series about $x = 0$ that converges for all $x$.

***Example* 9** Can we determine a series solution about $x = 0$ for the differential equation

$$y'' + (\sin x)y' + (1 + x^2)y = 0,$$

and if so, what is the radius of convergence?

*Solution*

For this differential equation, $p(x) = \sin x$ and $q(x) = 1 + x^2$. Recall from calculus that $\sin x$ has a Taylor series expansion about $x = 0$ that converges for all $x$. Further, $q$ also has a Taylor series expansion about $x = 0$, namely, $q(x) = 1 + x^2$, that converges for all $x$. Thus there is a series solution of the form $y = \sum_{n=0}^{\infty} a_n x^n$ with $a_0$ and $a_1$ arbitrary, and the series converges for all $x$.

Chapter 17

# Laplace and Inverse Laplace Transforms

## 17.1 Introduction

The Laplace[1] transformation is a powerful method for solving linear differential equations arising in engineering and mathematics. In this method the given differential equation is transformed into an algebraic equation (called subsidiary equation) and is solved purely by algebraic manipulations. Finally the solution of the subsidiary equation is transformed back to get the required solution of the original differential equation. Hence the place Laplace transformation reduces the problem of solving a differential equa-

[1] French mathematician Pierre Simon De Laplace (1749-1827) made important contributions to celestial mechanics and probability theory.

tion to an algebraic problem.

Before discussing the possible existence of $\int_a^\infty f(t)\,dt$, it is helpful to define certain terms. A function $f$ is said to be **piecewise continuous** on an interval $\alpha \le t \le \beta$ if the interval can be partitioned by a finite number of points $\alpha = t_0 < t_1 < \cdots < t_n = \beta$ so that

1. $f$ is continuous on each open subinterval $t_{i-1} < t < t_i$.
2. $f$ approaches a finite limit as the endpoints of each subinterval are approached from within the subinterval.

In other words, $f$ is piecewise continuous on $\alpha \le t \le \beta$ if it is continuous there except for a finite number of jump discontinuities. If $f$ is piecewise continuous on $\alpha \le t \le \beta$ for every $\beta > \alpha$, then $f$ is said to be piecewise continuous on $t \ge \alpha$. An example of a piecewise continuous function is shown in Fig.17.1.

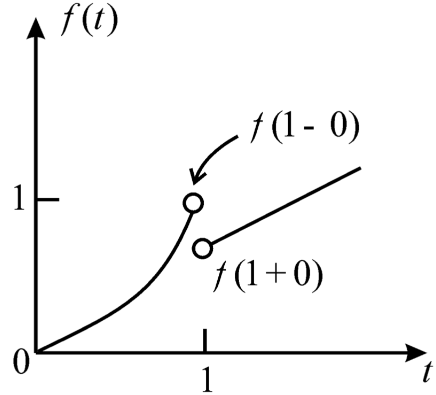


Figure 17.1:

The integral of a piecewise continuous function on a finite interval is just the sum of the integrals on the subintervals created by the partition points. For instance, for the function $f(t)$ shown in Fig.17.1, we have

$$\int_{\alpha}^{\beta} f(t)dt = \int_{\alpha}^{t_1} f(t)dt + \int_{t_1}^{\beta} f(t)dt \tag{17.1}$$

For the function shown in Fig. 17.1 we have assigned values to the function at the endpoints $\alpha$ and $\beta$ , and at the partition point$t_1$. However, as far as the integrals in Eq.(17.1) are concerned, it does not matter whether $f(t)$ is defined at these points, or what values may be assigned to $f(t)$ at these points. The values of the integrals in Eq.(17.1) remain the same regardless.

Thus, if $f$ is piecewise continuous on the interval$a \leq t \leq A$, then $\int_a^A f(t)\,dt$ exists. Hence, if $f$ is piecewise continuous for $t \geq a$, then $\int_a^A f(t)\,dt$ exists for each $A > a$. However, piecewise continuity is not enough to ensure convergence of the improper integral $\int_a^\infty f(t)\,dt$.

If $f$ cannot be integrated easily in terms of elementary functions, the definition of convergence of $\int_a^\infty f(t)\,dt$ may be difficult to apply. Frequently, the most convenient way to test the convergence or divergence of an improper integral is by the following comparison theorem which is analogous to a similar theorem for infinite series.

**Theorem A (Comparison Theorem)** If $f$ is piecewise continuous for $t \geq a$, if $|f(t)| \leq g(t)$ when $t \geq M$ for some positive constant $M$, and if $\int_M^\infty g(t)\,dt$ converges, then $\int_a^\infty f(t)\,dt$ also converges. On the other hand, if $f(t) \geq g(t) \geq 0$ for $t \geq M$, and if $\int_M^\infty g(t)\,dt$ diverges, then $\int_a^\infty f(t)\,dt$ also diverges.

**Integral Transforms**

Among the tools that are very useful for solving linear differential equations are **integral transforms.** An *integral transform* is a relation of the form

$$F(s) = \int_\alpha^\beta K(s,t)\,f(t)\,dt \qquad (17.2)$$

where $K(s,t)$ is a given function, called the **kernel** of the transformation, and the limits of integration $\alpha$ and $\beta$ are also given. It is possible that $\alpha = -\infty$ or $\beta = \infty$, or both. The relation (17.2) transforms the function $f$ into another function $F$, which is called the **transform** of $f$. In the next definition we see that Laplace transform makes use of the kernel $K(s,t) = e^{-st}$.

## 17.2 The Laplace Transform

**Definition** Let $f(t)$ be a given function which is defined for all t $\geq 0$. We multiply $f(t)$ by $e^{-st}$ and integrate with respect to $t$ from 0 to $\infty$, then, if the resulting integral exists, it is a function of s, denoted by $F(s)$ and defined by

$$F(s) = \int_0^\infty e^{-st} f(t) dt.$$

The function $F(s)$ is called the place Laplace transformation of the original function. In symbols we write

$$\mathrm{L}[f(t)] = \int_0^\infty e^{-st} f(t)dt = F(s)$$

where $s$ is a real or complex parameter.

**Remarks**

1. If the integral $\int_0^\infty e^{-st} f(t)dt$ converges for some value of $s$, then the Laplace transform of $f(t)$ is said to exist, otherwise it does not exist.

2. The operator L that transforms $f(t)$ into $F(s)$ is called the **Laplace transform operator**.

**Notation** Original functions are denoted by lowercase letters and their Laplace transforms by the same letters in capitals, so that $F(s)$ denotes the transform of $f(t)$, and $Y(s)$ denote the transform of $y(t)$, and so on.

***Example* 1** Find the Laplace transform of the following functions

(a) 0 (b) 1 (c) $t$ (d) $e^{at}$ (e) $e^{-at}$ (f) $e^{i\,at}$

*Solution*

(a) Here $f(t) = 0$. Hence

$$\mathrm{L}[0] = \int_0^\infty e^{-st} \cdot 0dt = \int_0^\infty 0dt = 0.$$

(b) $\mathrm{L}[1] = \int_0^\infty e^{-st} \cdot 1\,dt = \int_0^\infty e^{-st}dt$

Now the interval of integration of $\int_0^\infty e^{-st}dt$ is infinite and hence the integral is an **improper integral.** We first evaluate it.

$$\begin{aligned}
\int_0^\infty e^{-st}dt &= \lim_{b\to\infty}\int_0^b e^{-st}dt \\
&= \lim_{b\to\infty}\left[\frac{e^{-st}}{-s}\right]_0^b \\
&= \frac{1}{-s}\lim_{b\to\infty}\left(e^{-sb}-e^0\right) \\
&= \frac{1}{-s}(0-1) \text{ for } s>0, \text{ since} \\
&\quad \lim_{b\to\infty} e^{-sb} = 0, \quad \text{for } s>0 \\
&= \frac{1}{s}, \text{ if } s>0
\end{aligned}$$

Hence

$$\mathrm{L}[1] = \frac{1}{s} \text{ for } s>0.$$

(c) $\mathrm{L}[t] = \int_0^\infty e^{-st}\cdot t\ dt = \int_0^\infty t\,e^{-st}dt$

Now we evaluate the improper integral $\int_0^\infty t\,e^{-st}dt$ as follows:

$$\int_0^\infty te^{-st}dt = \lim_{b\to\infty}\int_0^b te^{-st}dt = \lim_{b\to\infty}\left\{\left[t\frac{e^{-st}}{-s}\right]_0^b - \int_0^b 1.\frac{e^{-st}}{-s}dt\right\} \tag{17.3}$$

Now

$$\begin{aligned}
\lim_{b\to\infty}\left[t\frac{e^{-st}}{-s}\right]_0^b &= \frac{1}{-s}\lim_{b\to\infty}\left\{be^{-sb}-0\right\} \\
&= \frac{1}{-s}\lim_{b\to\infty}\frac{b}{e^{sb}} \quad (\frac{\infty}{\infty}\text{ form}) \\
&= \frac{1}{-s}\lim_{b\to\infty}\frac{1}{se^{sb}},\ \text{treating } b \text{ as variable,} \\
&\quad s \text{ as constant and applying L}'\text{Hospital Rule} \\
&= 0,\ \text{for } s>0 \text{ since } \lim_{b\to\infty} e^{-sb}=0
\end{aligned}$$

Also, by part (b) above

$$\lim_{b\to\infty}\int_0^b e^{-st}dt = \int_0^\infty e^{-st}dt = \frac{1}{s} \text{ for } s>0.$$

Substituting these values in (17.3), we obtain

$$\begin{aligned}
L(t) &= 0-\frac{1}{-s}\times\frac{1}{s} \\
&= \frac{1}{s^2},\ \text{if } s>0.
\end{aligned}$$

(d)

$$\begin{aligned}
\mathrm{L}[e^{at}] &= \int_0^\infty e^{-st}e^{at}dt \\
&= \int_0^\infty e^{-(s-a)t}dt \\
&= \lim_{b\to\infty}\int_0^b e^{-(s-a)t}dt
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{b\to\infty}\left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^b \\
&= \frac{1}{s-a},\ \text{if } s-a>0, \\
&\quad \text{since then } \lim_{b\to\infty} e^{-(s-a)b}=0
\end{aligned}
$$

(e)

$$
\begin{aligned}
\mathrm{L}[e^{-at}] &= \int_0^\infty e^{-st}e^{-at}dt \\
&= \int_0^\infty e^{-(s+a)t}dt = \lim_{b\to\infty}\int_0^b e^{-(s+a)t}dt \\
&= \lim_{b\to\infty}\left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^b \\
&= \frac{1}{s+a},\ \text{if } s+a>0,\ \text{since then } \lim_{b\to\infty} e^{-(s+a)b}=0.
\end{aligned}
$$

*Aliter:* (e) can also be obtained from (d) directly by replacing $a$ by $-a$.

(f)

$$
\begin{aligned}
\mathrm{L}[e^{iat}] &= \int_0^\infty e^{-st}e^{iat}dt = \int_0^\infty e^{-(s-ia)t}dt \\
&= \lim_{b\to\infty}\int_0^b e^{-(s-ia)t}dt = \lim_{b\to\infty}\left[\frac{e^{-(s-ia)t}}{-(s-ia)}\right]_0^b
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{-(s-ia)} \lim_{b\to\infty} \left[e^{-st}e^{i\,at}\right]_0^b \\
&= \frac{1}{-(s-ia)} \lim_{b\to\infty} \left(e^{-sb}e^{i\,ab} - 1\right) \\
&= \frac{1}{-(s-ia)} \left(\lim_{b\to\infty} e^{-sb} \lim_{b\to\infty} e^{i\,ab} - 1\right) \\
&= \frac{1}{-(s-ia)}(0 \;-\; 1), \text{ since } \lim_{b\to\infty} e^{-sb} = 0 \\
&= \frac{1}{s-ia}, \text{ if } s > 0.
\end{aligned}
$$

***Example* 2** When $n$ is a positive integer, find a reduction formula for $\mathrm{L}[t^n]$and hence evaluate $\mathrm{L}[t^n]$.

*Solution*

By the definition of Laplace transformation,

$$\mathrm{L}[t^n] = \int_0^\infty e^{-st}t^n dt = \int_0^\infty t^n e^{-st} dt = \lim_{b\to\infty} \int_0^b t^n e^{-st} dt. \quad (17.4)$$

Now take $u = t^n,\; v' = e^{-st}$ in the integration by parts formula

$$\int uv' = uv - \int u'v$$

and obtain

$$\int_0^b t^n e^{-st} dt = \left[\frac{t^n e^{-st}}{-s}\right]_0^b - \int_0^b nt^{n-1}\frac{e^{-st}}{-s} dt.$$

Hence

$$\lim_{b\to\infty}\int_0^b t^n e^{-st}dt = \frac{1}{-s}\lim_{b\to\infty}\left[t^n e^{-st}\right]_0^b + \frac{n}{s}\lim_{b\to\infty}\int_0^b t^{n-1}e^{-st}dt \tag{17.5}$$

Now

$$\begin{aligned}
\lim_{b\to\infty}\left[t^n e^{-st}\right]_0^b &= \lim_{b\to\infty} b^n e^{-sb} \\
&= \lim_{b\to\infty}\frac{b^n}{e^{sb}}\left(\frac{\infty}{\infty}\text{ form}\right) \\
&\overset{L}{=} \lim_{b\to\infty}\frac{n\,b^{n-1}}{s\,e^{sb}},\ \text{treating } b \text{ as variable, } s \text{ as constant} \\
&\quad \text{and applying L}'\text{Hopital}'\text{s Rule} \\
&= \frac{n}{s}\lim_{b\to\infty}\frac{b^{n-1}}{e^{sb}}\left(\text{ again in }\frac{\infty}{\infty}\text{ form}\right) \\
&\overset{L}{=} \frac{n}{s}\lim_{b\to\infty}\frac{(n-1)\,b^{n-2}}{s\,e^{sb}}\left(\text{ again in }\frac{\infty}{\infty}\text{ form}\right) \\
&\overset{L}{=} \frac{n(n-1)}{s^2}\lim_{b\to\infty}\frac{(n-2)\,b^{n-3}}{s\,e^{sb}}\quad\left(\text{again in }\frac{\infty}{\infty}\text{ form }\right) \\
&\vdots \\
&\overset{L}{=} \frac{n!}{s^n}\lim_{b\to\infty}\frac{1}{e^{sb}} \\
&= 0.
\end{aligned}$$

Substituting this in (17.5) and noting that

$$\lim_{b\to\infty}\int_0^b t^{n-1}e^{-st}dt = \int_0^\infty t^{n-1}e^{-st}dt = L[t^{n-1}],$$

we obtain that

$\mathrm{L}[t^n] = \frac{n}{s}\mathrm{L}[t^{n-1}]$ , if $s > 0$.

Hence the required reduction formula is

$$\mathrm{L}[t^n] = \frac{n}{s}\mathrm{L}[t^{n-1}] \tag{17.6}$$

Evaluation of $\mathbf{L}[t^n]$:

$$\mathrm{L}[t^n] = \frac{n}{s}\mathrm{L}[t^{n-1}]$$

$= \frac{n}{s} \cdot \frac{n-1}{s}\mathrm{L}[t^{n-2}]$, using the reduction

formula (17.6) with $n$ replaced by $n-1$

$$= \frac{n}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s}\mathrm{L}[t^{n-3}]$$

. . . . . . . . . .

$$= \frac{n}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s} \cdot \ \ldots \ \cdot \frac{2}{s} \cdot \frac{1}{s}\mathrm{L}[t^0]$$

$$= \frac{n}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s} \cdot \ \ldots \ \cdot \frac{2}{s} \cdot \frac{1}{s} \cdot \frac{1}{s},$$

since $L[t^0] = L[1] = \frac{1}{s}$

$$= \frac{n\,(n-1)\,(n-2) \cdot \ \ldots \ \cdot 2 \cdot 1}{s^{n+1}}$$

$$= \frac{n!}{s^{n+1}}$$

***Example*** **3** Using the definition of Laplace transformation prove that

$$L[t^a] = \frac{\Gamma(a+1)}{s^{a+1}},$$

for any positive number $a$ (*need not be positive integer*). Hence deduce that

$$L[t^a] = \frac{a\,!}{s^{a+1}},$$

when $a$ is a positive integer.

**Note** $\Gamma$ seen above is the gamma function defined for $a > 0$ by

$$\Gamma(a) = \int_0^\infty e^{-x} x^{a-1} dx\,, \quad \text{where } a > 0$$

*Solution*

By the definition of Laplace transformation we have

$$L[t^a] = \int_0^\infty e^{-st} t^a dt.$$

Now, let $st = x$, so that $t = \frac{x}{s}$ and $dt = \frac{dx}{s}$. Also, when $t = 0, x = 0$ and when $t = \infty, x = \infty$. Hence, if $s > 0$, we have

$$L[t^a] = \int_0^\infty e^{-st} t^a dt = \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^a \frac{dx}{s}$$

$$= \frac{1}{s^{a+1}} \int_0^\infty e^{-x} x^a dx = \frac{\Gamma\,(a+1)}{s^{a+1}}.$$

i.e., $L[t^a] = \frac{\Gamma(a+1)}{s^{a+1}}, \quad \text{if} \quad s > 0.$

When $a$ is a positive integer, $\Gamma(a+1) = a!$.

Hence in that case $L[t^a] = \frac{a!}{s^{a+1}}$

**Theorem 1 (Linearity of the Laplace Transformation)**

**The Laplace transformation is a linear transformation.**

i.e. if $a$ and $b$ are any two constants and $f\ (t)$ and $g\ (t)$ are functions with Laplace transforms $F\ (s)$ and $G\ (s)$ respectively, then

$$\mathrm{L}[af(t)+bg(t)] = a\mathrm{L}[f(t)] + b\mathrm{L}[g(t)] = aF(s)+bG(s)$$

In particular, if k is a constant, $\mathrm{L}[kf(t)] = k\mathrm{L}[f(t)] = kF(s)$.

**Proof** By the definition

$$\mathrm{L}[af(t)+bg(t)] = \int_0^\infty e^{-st}[af(t)+bg(t)]dt$$

$$= a\int_0^\infty e^{-st}f(t)dt + b\int_0^\infty e^{-st}g(t)dt$$

$$= a\mathrm{L}[f(t)] + b\mathrm{L}[g(t)]$$

$$= aF(s)+bG(s).$$

***Example*** **4** Find the Laplace transform of

(a) cos $at$ (b) sin $at$

*Solution*

Recall the Euler formula $e^{i\theta} = \cos\theta + i\ \sin\theta$. Putting $\theta = at$, we obtain

$$e^{iat} = \cos at + i\ \sin at$$

Hence by the Linearity of the Laplace transformation (Theorem

1), we have

$$\mathrm{L}[e^{iat}] = \mathrm{L}[\cos at] + i\ \mathrm{L}[\sin at] \tag{17.7}$$

We also know that

$$\mathrm{L}\left[e^{iat}\right] = \frac{1}{s-ia} = \frac{s+ia}{(s-ia)(s+ia)} = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2} \tag{17.8}$$

Equating the real and imaginary parts of (17.7) and (17.8), we obtain

$\mathrm{L}\ [\cos\ at] = \frac{s}{s^2+a^2}$ and $\mathrm{L}\ [\sin at] = \frac{a}{s^2+a^2}$.

**Aliter:** The Laplace transform of $\cos at$ and $\sin at$ can be obtained directly from the definition of Laplace transform as follows:

By the definition, $L(\cos at) = \int_0^\infty e^{-st}\cos atdt$.

Take $I = \int_0^\infty e^{-st}\cos atdt$

Then

$$I = \int_0^\infty e^{-st}\cos at\,dt = \lim_{A\to\infty}\int_0^A e^{-st}\cos\ at\,dt$$

$$= \lim_{A\to\infty}\left\{\left[e^{-st}\frac{\sin at}{a}\right]_0^A - \int_0^A (-s)\,e^{-st}\frac{\sin at}{a}dt\right\}$$

$= \lim\limits_{A\to\infty}\frac{s}{a}\int_0^A e^{-st}\sin atdt$, if $s > 0$ [Here we have used the fact that, since $-1 \le \sin aA \le 1 \Rightarrow -e^{-sA} \le e^{-sA}\sin aA \le e^{-sA}$ and $\lim\limits_{A\to\infty} e^{-sA} = 0$, by the Sandwich Theorem, $\lim\limits_{A\to\infty} e^{-sA}\sin aA = 0$

and hence $\lim\limits_{A\to\infty} e^{-sA}\frac{\sin aA}{a} = 0]$

$$= \lim_{A\to\infty}\frac{s}{a}\left\{\left[e^{-st}\left(\frac{-\cos at}{a}\right)\right]_0^A - \int_0^A (-s)\, e^{-st}\left(\frac{-\cos at}{a}\right) dt\right\}$$

$$= \frac{s}{a}\left\{\frac{1}{a} + \frac{s}{a}\int_0^\infty e^{-st}\cos\, at\; dt\right\}$$

$= \frac{s}{a}\left\{\frac{1}{a} + \frac{s}{a}I\right\} = \frac{s}{a^2} - \frac{s^2}{a^2}I$, if $s > 0$.

$\therefore \left(1 + \frac{s^2}{a^2}\right) I = \frac{s}{a^2}$, if $s > 0$.

i.e., $I = \frac{s}{s^2+a^2}$, if $s > 0$.

i.e., $L(\cos at) = \frac{s}{s^2+a^2}$, if $s > 0$.

Proceeding similarly, it can be seen that

$L(\sin at) = \frac{a}{s^2+a^2}$, if $s > 0$.

***Example*** **5** Find the Laplace transform of

1. cosh $at$
2. sinh $at$

*Solution*

1. $\mathrm{L}[\cosh at] = \mathrm{L}\left[\frac{1}{2}\left(e^{at} + e^{-at}\right)\right] = \frac{1}{2}\mathrm{L}\left[e^{at}\right] + \frac{1}{2}\mathrm{L}\left[e^{-at}\right]$,

by the linearity of the Laplace transformation

$$= \frac{1}{2}\left[\frac{1}{s-a}\right] + \frac{1}{2}\left[\frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{s+a+s-a}{s^2-a^2}\right] = \frac{s}{s^2-a^2}$$

2. Since $\sinh at = \frac{e^{at}-e^{-at}}{2}$, we have

$$\mathrm{L}[\sinh at] = \mathrm{L}\left[\frac{1}{2}\left(e^{at} - e^{-at}\right)\right] = \frac{1}{2}\mathrm{L}\left[e^{at}\right] - \frac{1}{2}\mathrm{L}\left[e^{-at}\right],$$

by the linearity of the Laplace transformation

$$= \frac{1}{2}\left[\frac{1}{s-a}\right] - \frac{1}{2}\left[\frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{s+a-s+a}{s^2-a^2}\right] = \frac{a}{s^2-a^2}.$$

**Theorem 2 (Existence theorem for Laplace transforms)** Suppose that

(i). $f(t)$ is piecewise continuous on the finite interval $0 \le t \le A$ for any positive $A$.

(ii). there exist real constants $K$, $a$, and $M$, with $K$ and $M$ positive, such that

$$|f(t)| \le Ke^{at} \text{ when } t \ge M.$$

Then the Laplace transform of $f(t)$ exists for all $s > a$.

**Proof** We have to show that $\int_0^\infty e^{-st} f(t)\,dt$ converges for $s > a$.

$$\int_0^\infty e^{-st} f(t)\,dt = \int_0^M e^{-st} f(t)\,dt + \int_M^\infty e^{-st} f(t)\,dt. \qquad (17.9)$$

The first integral on the right side of Eq. (17.9) exists since $f$ is piecewise continuous on the interval $0 \le t \le A$ for any positive $A$ (and in particular for the interval $0 \le t \le M$).

Also, for $t \geq M$,

$$\left|e^{-st} f(t)\right| \leq \left|e^{-st}\right| \, |f(t)| \, = \, e^{-st} \, |f(t)|$$

$$\leq \, K \cdot e^{-st} \cdot e^{at} = K e^{(a-s)t}$$

Hence, by Theorem A, $\int_M^\infty e^{-st} f(t)\, dt$ converges provided $\int_M^\infty e^{(a-s)t}\, dt$ converges. Now, we have

$$\int_M^\infty e^{(a-s)t}\, dt = \lim_{A\to\infty} \int_M^A e^{(a-s)t} dt$$

$$= \lim_{A\to\infty} \left[ \frac{e^{(a-s)t}}{a-s} \right]_M^A$$

$$= \lim_{A\to\infty} \left[ \frac{e^{(a-s)A} - e^{(a-s)M}}{a-s} \right]$$

Since $e^{(a-s)A} \to 0$ as $A \to \infty$ only when $a - s < 0$, the above implies $\int_M^\infty e^{(a-s)t}\, dt = -\frac{e^{(a-s)M}}{a-s}$, if $a - s < 0$.

Hence $\int_M^\infty e^{(a-s)t} dt$ converges; hence $\int_M^\infty e^{-st} f(t)\, dt$ converges if $a - s < 0$.

Hence the proof of the theorem.

**Remark** The conditions in the theorem are not necessary, as the example $f(t) = t^{-1/2}$ shows. This function has an infinite discontinuity at $t = 0$, so it is not piecewise continuous, but its integral from 0 to $b$ exists; and since it is bounded for large $t$, its Laplace transform exists. Indeed, for $s > 0$ we have

$$L\,[t^{-1/2}] = \int_0^\infty e^{-st} t^{-1/2}\, dt,$$

and the change of variable $st = u$ gives

$$L\,[t^{-1/2}] = s^{-1/2} \int_0^\infty e^{-u}\, u^{-1/2}\; du.$$

Another change of variable, $u = w^2$, leads to

$$L[t^{-1/2}] = 2s^{-1/2} \int_0^\infty e^{-w^2}\, dw.$$

Noting that $\int_0^\infty e^{-w^2}\, dw = \sqrt{\pi/2}$, the above gives

$$L[t^{-1/2}] = \sqrt{\frac{\pi}{s}}.$$

***Example* 6** Show that $\int_0^\infty e^{-w^2}\, dw = \sqrt{\pi/2}$.

*Solution*

Take $I = \int_0^\infty e^{-w^2}\, dw$. Then $w$ being a dummy variable, we can write

$$I^2 = \left(\int_0^\infty e^{-x^2}\, dx\right)\left(\int_0^\infty e^{-y^2}\, dy\right) = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy.$$

The evaluation of the double integral is possible by changing to polar coordinates $[x^2 + y^2 = r^2,\; dx\, dy = r\, dr\, d\theta, 0 \le r < \infty,$

$0 \leq \theta < 2\pi]$, and hence we obtain

$$I^2 = \frac{\pi}{2}.$$

Hence

$$\int_0^\infty e^{-w^2}\,dw = I = \sqrt{\frac{\pi}{2}}.$$

**Exercises**

Find the Laplace transform of the following functions

1. $1 - e^{-t}$
2. $\cosh 2t - 1$
3. $2e^{-at} - 1$
4. $1 + t - \cos t + \sin t$
5. $\frac{1-\cos 2t}{4}$
6. $\cosh 2t - 1$
7. $\sinh 2t - t$
8. $\frac{1}{4}(1 - 2t + \sin 2t - \cos 2t)$
9. $\frac{e^{2t}-1-2t-2t^2}{8}$
10. $\frac{t^2-2\pi t-2+2e^{\pi t}}{2\pi^2}$
11. $\frac{t^3}{6\pi^2} - \frac{t}{\pi^4} + \frac{\sin \pi t}{\pi^5}$

**Answers**

1. $\frac{1}{s^2+s}$
2. $\frac{4}{s^3-4s}$
3. $\frac{1}{s}\left(\frac{s-a}{s+a}\right)$
4. $\frac{2s^2+s+1}{s^2(s^2+1)}$
5. $\frac{1}{s^3+4s}$
6. $\frac{1}{s^2}\left(\frac{s-a}{s+a}\right)$
7. $\frac{8}{s^4-4s^2}$
8. $\frac{1}{s^2}\left(\frac{s-2}{s^2+4}\right)$
9. $\frac{1}{s^4-2s^3}$
10. $\frac{s-\pi+\pi^2 s}{\pi^2 s^3(s-\pi)}$

11. $\frac{1}{s^4(s^2+\pi^2)}$

## 17.3 Inverse Laplace transform

**Definition** If $\mathrm{L}[f(t)] = F(s)$, then $f(t)$ is called an **inverse Laplace transform** of $F(s)$. Symbolically,

$$f(t) = \mathrm{L}^{-1}[F(s)].$$

In other words, the inverse transform of a given function $F(s)$ is that function $f(t)$ whose Laplace transform is $F(s)$.
For example, $\mathrm{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at}$, since $\mathrm{L}[e^{at}] = \frac{1}{s-a}$.
**Theorem 3(Linearity of the Inverse Laplace Transformation)**
If $a$ and $b$ are any constants $F(s)$ and $G(s)$ are functions with inverse Laplace transforms $f(t)$ and $g(t)$ respectively, then

$$\mathrm{L}^{-1}[aF(s) + bG(s)] = a\mathrm{L}^{-1}[F(s)] + b\mathrm{L}^{-1}[G(s)] = af(t) + bg(t).$$

***Example* 7** Find the inverse Laplace transform of:

1. $\frac{5}{s+3}$
2. $.\frac{2\pi}{s+\pi}$
3. $\frac{1}{s^2+25}$
4. $\frac{12}{s^2+16}$
5. $\frac{s}{s^2+25}$
6. $\frac{1}{s^4}$
7. $\frac{1}{s^2+81}$
8. $\frac{13}{s^2+25}$
9. $\frac{s}{s^2+36}$
10. $\frac{1}{s^5}$
11. $\frac{2}{s^2+4^2}$

*Solution*

1. $\mathrm{L}^{-1}[\frac{5}{s+3}] = 5\mathrm{L}^{-1}[\frac{1}{s+3}]$, by the Linearity of the inverse Laplace transformation

   $= 5e^{-3t}$, since $L\left(e^{-3t}\right) = \frac{1}{s+3}$

2. $\mathrm{L}^{-1}[\frac{2\pi}{s+\pi}] = 2\pi\mathrm{L}^{-1}[\frac{1}{s+\pi}] = 2\pi e^{-\pi t}$,

3. $\mathrm{L}^{-1}[\frac{1}{s^2+25}] = \frac{1}{5}\mathrm{L}^{-1}[\frac{5}{s^2+25}] = \frac{1}{5}\sin 5t$

4. $\mathrm{L}^{-1}[\frac{12}{s^2+16}] = 3\mathrm{L}^{-1}[\frac{4}{s^2+4^2}] = 3\sin 4t$

5. $\mathrm{L}^{-1}[\frac{s}{s^2+25}] = \mathrm{L}^{-1}[\frac{s}{s^2+5^2}] = \cos 5t$

6. $\mathrm{L}^{-1}[\frac{1}{s^4}] = \frac{t^3}{3!} = \frac{t^3}{6}$, since $\mathrm{L}\left[t^3\right] = \frac{3!}{s^4}$

7. $\mathrm{L}^{-1}[\frac{1}{s^2+81}] = \frac{1}{9}\mathrm{L}^{-1}[\frac{9}{s^2+9^2}] = \frac{1}{9}\sin 9t$

8. $\mathrm{L}^{-1}[\frac{13}{s^2+25}] = \frac{13}{5}\mathrm{L}^{-1}[\frac{5}{s^2+5^2}] = \frac{13}{5}\sin 5t$

9. $\mathrm{L}^{-1}[\frac{s}{s^2+36}] = \mathrm{L}^{-1}[\frac{s}{s^2+6^2}] = \cos 6t$

10. $\mathrm{L}^{-1}[\frac{1}{s^5}] = \frac{t^4}{4!}$, since $\mathrm{L}\left[t^4\right] = \frac{4!}{s^5} = \frac{t^4}{24}$

11. $\mathrm{L}^{-1}[\frac{2}{s^2+4^2}] = \frac{1}{2}\mathrm{L}^{-1}[\frac{4}{s^2+4^2}] = \frac{1}{2}\sin 4t$

***Example* 8** Find the inverse Laplace transform of:

1. $\frac{s+1}{s^2+1}$
2. $\frac{5s}{s^2+36}$
3. $\frac{a_1}{s} + \frac{a_2}{s^2} + \frac{a_3}{s^3}$
4. $\frac{b_1}{s} + \frac{b_2}{s^2} + \frac{b_3}{s^3} + \frac{b_4}{s^4}$

*Solution*

1. $L^{-1}\left[\frac{s+1}{s^2+1}\right] = L^{-1}\left[\frac{s}{s^2+1}\right] + L^{-1}\left[\frac{1}{s^2+1}\right]$, using the linearity of the inverse Laplace transformation with $F(s) = \frac{s}{s^2+1}$, $G(s) = \frac{1}{s^2+1}$

$$= \cos t + \sin t$$

2. $\mathrm{L}^{-1}[\frac{5s}{s^2+36}] = 5\mathrm{L}^{-1}[\frac{s}{s^2+6^2}] = 5\cos 6t$

3. $\mathrm{L}^{-1}[\frac{a_1}{s} + \frac{a_2}{s^2} + \frac{a_3}{s^3}] = a_1\mathrm{L}^{-1}[\frac{1}{s}] + a_2\mathrm{L}^{-1}[\frac{1}{s^2}] + a_3\mathrm{L}^{-1}[\frac{1}{s^3}]$

$$= a_1 \cdot 1 + a_2 t + a_3\frac{t^2}{2!} = a_1 + a_2 t + a_3\frac{t^2}{2}.$$

4. $L^{-1}\left[\frac{b_1}{s} + \frac{b_2}{s^2} + \frac{b_3}{s^3} + \frac{b_4}{s^4}\right] = b_1 L^{-1}\left[\frac{1}{s}\right] + b_2 L^{-1}\left[\frac{1}{s^2}\right]$

$$+b_3 L^{-1}\left[\frac{1}{s^3}\right] + b_4 L^{-1}\left[\frac{1}{s^4}\right]$$

$$= b_1 + b_2 t + \frac{t^2}{2!}b_3 + \frac{t^3}{3!}b_4$$

**Method of Partial Fractions**

We illustrate the method of partial fraction for finding inverse Laplace transform through examples.

**Case1. Unrepeated Factor** $s - a$

***Example* 9** Find the inverse transform of $\frac{s+1}{s(s^2+s-6)}$

*Solution*

Take $F(s) = \frac{s+1}{s(s-2)(s+3)} = \frac{A_1}{s} + \frac{A_2}{s-2} + \frac{A_3}{s+3}$.

Determination of $A_1, A_2,$ and $A_3$:

Multiplication by the common denominator $s(s-2)\,(s+3)$ gives

$$s+1=(s-2)(s+3)\,A_1+s(s+3)\,A_2+s(s-2)A_3.$$

Taking $s=0,\ \ s=2,\ \ s=-3$, we obtain, respectively,

$$1=-2\cdot 3A_1;3=2\cdot 5A_2;-2=-3(-5)A_3.$$

Hence $A_1=-1/6,\ A_2=3/10,$ and $A_3=-2/15.$ Therefore,

$$y(t)=L^{-1}[F(s)]=-\frac{1}{6}+\frac{3}{10}e^{2t}-\frac{2}{15}e^{-3t}.$$

***Example* 10** Find the inverse transform of $\frac{1}{s(s+1)(s+2)}$

*Solution*

Let

$$\frac{1}{s(s+1)(s+2)}=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+2}$$

which implies

$1 = A(s+1)\ (s+2) + Bs(s+2) + Cs(s+1).$

Putting $s$ = 0, we get $A = \frac{1}{2}$, putting $s$ = -1, we get $B$ = -1 and by putting $s$ = -2, we get $C = \frac{1}{2}$.

$$\mathrm{L}^{-1}(\frac{1}{s(s+1)(s+2)})=\mathrm{L}^{-1}(\frac{1/2}{s}-\frac{1}{(s+1)}+\frac{1/2}{(s+2)})$$

$$=\frac{1}{2}\mathrm{L}^{-1}(\frac{1}{s})-\mathrm{L}^{-1}(\frac{1}{s+1})+\frac{1}{2}\mathrm{L}^{-1}(\frac{1}{s+2})$$

$$=\frac{1}{2}-e^{-t}+\frac{1}{2}e^{-2t}$$

**Case2. Repeated Factor** $(s-a)^m$

We recall that repeated factors$(s-a)^2$, $(s-a)^3$, etc., require partial fractions

$\frac{A_2}{(s-a)^2}+\frac{A_1}{s-a}$, $\quad \frac{A_3}{(s-a)^3}+\frac{A_2}{(s-a)^2}+\frac{A_1}{s-a}$, etc

respectively.

***Example* 11** Find the inverse transform of $\frac{s^3-4s^2+4}{s^2(s^2-3s+2)}$.

*Solution*

Take $F(s)=\frac{s^3-4s^2+4}{s^2(s^2-3s+2)}$.

It can be written as

$$F(s)=\frac{s^3-4s^2+4}{s^2(s-2)(s-1)}=\frac{A_2}{s^2}+\frac{A_1}{s}+\frac{B}{s-2}+\frac{C}{s-1}$$

Multiplication with $s^2(s-2)(s-1)$ gives

$$\begin{aligned} s^3-4s^2+4 &= A_2(s-2)(s-1)+A_1s(s-2)(s-1) \\ &\quad +Bs^2(s-1)+Cs^2(s-2) \end{aligned} \tag{17.10}$$

For $s=1$ this is $1=C(-1)$, hence $C=-1$.

For $s=2$ it is $-4=4B$, hence $B=-1$.

For $s=0$ we get $4=2A_2$, hence $A_2=2$.

Differentiation of (17.10) gives

$$3s^2-8s=A_2(2s-3)+A_1(s-2)(s-1)$$

+ further terms all containing a factor $s$.

For $s=0$ this is $0=-3A_2+2A_1$. Hence $A_1=3A_2/2=3$.

Hence

$$y(t) = L^{-1}[F(s)] = L^{-1}\left\{\frac{2}{s^2} + \frac{3}{s} - \frac{1}{s-2} - \frac{1}{s-1}\right\}$$

$$= 2t + 3 - e^{2t} - e^t.$$

**Case 3. Unrepeated Complex Factors** $(s - \mathrm{a})(s - \bar{\mathrm{a}})$

If $s - a$ with complex $a = \alpha + i\beta$ is a factor of $G(s)$, so is $s - \bar{a}$ with $\bar{a} = \alpha - i\beta$ the conjugate. To $(s-a)(s-\bar{a}) = (s-\alpha)^2 + \beta^2$ there corresponds the partial fraction

$\frac{As+B}{(s-a)(s-\bar{a})}$ or $\frac{As+B}{(s-\alpha)^2+\beta^2}$

We encounter such a situation in the next example.

***Example* 12** Find $L^{-1}\left[\frac{3s+1}{(s-1)(s^2+1)}\right]$.

*Solution*

$$\frac{3s+1}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1}$$

implies

$$3s + 1 = A(s^2+1) + (Bs + C)(s-1).$$

Putting $s = 1$, we get $A = 2$, putting $s = 0$, we get $A - C = 1$ implies $C = 1$. Comparing coefficients of $s^2$ we get $A + B = 0$ or $B = -A = -2$. Hence

$$\frac{3s+1}{(s-1)(s^2+1)} = \frac{2}{s-1} + \frac{-2s+1}{s^2+1}$$

and

$$\mathrm{L}^{-1}(\frac{3s+1}{(s-1)(s^2+1)}) = \mathrm{L}^{-1}(\frac{2}{s-1} + \frac{-2s+1}{s^2+1})$$

$$= 2\mathrm{L}^{-1}(\frac{1}{s-1}) - 2\mathrm{L}^{-1}(\frac{s}{s^2+1}) + \mathrm{L}^{-1}(\frac{1}{s^2+1}) = 2e^t - 2\cos t + \sin t$$

***Example*** **13** Find the inverse transform of $\frac{1}{s^2}\left(\frac{s+1}{s^2+a}\right)$.

*Solution*

$$\frac{1}{s^2}\left(\frac{s+1}{s^2+a}\right) = \frac{As+B}{s^2} + \frac{Cs+D}{s^2+a}$$

(OR can be written as $\frac{1}{s^2}\left(\frac{s+1}{s^2+a}\right) = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+a}$).

This implies

$$s+1 = (As+B)(s^2+a) + (Cs+D)s^2.$$

Putting $s = 0$, we get $1 = Ba$, hence $B = \frac{1}{a}$.

Equating like powers of $s$, we obtain

$$A + C = 0, B + D = 0, Aa = 1,$$

and hence

$$A = \frac{1}{a}, D = -B = -\frac{1}{a}, C = -A = -\frac{1}{a}.$$

Hence

$$\frac{1}{s^2}\left(\frac{s+1}{s^2+a}\right) = \frac{1}{a}\left(\frac{s+1}{s^2} - \frac{s+1}{s^2+(\sqrt{a})^2}\right)$$

and

$$\mathrm{L}^{-1}\left[\frac{1}{s^2}\left(\frac{s+1}{s^2+a}\right)\right]$$

$$= \frac{1}{a}\left\{\mathrm{L}^{-1}\left[\frac{1}{s}\right] + \mathrm{L}^{-1}\left[\frac{1}{s^2}\right] - \mathrm{L}^{-1}\left[\frac{s}{s^2+(\sqrt{a})^2}\right] - \mathrm{L}^{-1}\left[\frac{1}{s^2+(\sqrt{a})^2}\right]\right\}$$

$$= \frac{1}{a}\left(1 + t - \cos\sqrt{a}\,t - \frac{\sin\sqrt{a}\,t}{\sqrt{a}}\right).$$

***Case 4. Repeated Complex Factors** $[(s-a)(s-\bar{a})]^2$

In this case the partial fractions are of the form

$$\frac{As+B}{[(s-a)(s-\bar{a})]^2} + \frac{Ms+N}{(s-a)(s-\bar{a})}.$$

A problem of this type will be discussed in a coming chapter.

**Exercises**

In Exercises 1-18, find $f(t)$ *if* $L[f(t)]$ is:

1. $\frac{1}{s^2+s}$
2. $\frac{1}{(s-a)(s-b)}$ $\quad (a \neq b)$
3. $\frac{4}{(s+1)(s+2)}$
4. $\frac{9}{s^2+3s}$
5. $\frac{4}{s^3-4s}$
6. $\frac{1}{s^2+s}$
7. $\frac{1}{s^2+4s}$
8. $\frac{8}{s^4-4s^2}$
9. $\frac{1}{s^2}\left(\frac{s-2}{s^2+4}\right)$
10. $\frac{3s+16}{s^2-s-6}$
11. $\frac{s-a}{s^2(s+a)}$
12. $\frac{1}{s}\left(\frac{s-a}{s+a}\right)$
13. $\frac{1}{s^3+4s}$
14. $\frac{1}{s^2}\left(\frac{s-a}{s+a}\right)$
15. $\frac{1}{s^4-2s^3}$
16. $\frac{2s-\pi}{s^3(s-\pi)}$
17. $\frac{1}{s^4(s^2+\pi^2)}$
18. $\frac{3s+7}{s^2-2s-3}$

19. $F(s) = \frac{5}{s^2+4}$

20. $F(s) = \frac{2}{s^2+3s-4}$

21. $F(s) = \frac{2s-3}{s^2-4}$

22. $F(s) = \frac{2s+1}{s^2-2s+2}$

23. $F(s) = \frac{3-4s}{s^2+4s+5}$

24. (*Finding Laplace transforms of certain functions using their Taylor series expansions*)

(a) Using the Taylor series for $\sin t$

$$\sin t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!}$$

and assuming that the Laplace transform of this series can be computed term by term, verify that

$$L\{\sin t\} = \frac{1}{s^2+1}, \quad s > 1.$$

(b) Let

$$f(t) = \begin{cases} (\sin t)/t & t \neq 0, \\ 1, & t = 0. \end{cases}$$

Find the Taylor series for $f$ about $t = 0$. Assuming that the Laplace transform of this function can be computed term by term, verify that

$$L\{f(t)\} = \arctan(1/s), \quad s > 1.$$

*Hint to Exercise 17:* $\frac{1}{s^4(s^2+\pi^2)} = \frac{As^3+Bs^2+Cs+D}{s^4} + \frac{Es+F}{s^2+\pi^2}$

Equating like powers of $s$ and simplifying, we get

$$A = C = E = 0, \;\; B = -\frac{1}{\pi^4}, \;\; D = \frac{1}{\pi^2}, \;\; F = \frac{1}{\pi^4}.$$

*Hint to Exercise 18:* $s^2 - 2s - 3 = (s-3)(s+1)$

**Answers**

1. $1 - e^{-t}$
2. $\frac{e^{at}-e^{bt}}{a-b}(a \neq b)$
3. $4\left(e^{-t} - e^{-2t}\right)$
4. $3\left(1 - e^{-3t}\right)$
5. $\cosh 2t - 1$ or $\frac{1}{2}e^{2t} + \frac{1}{2}e^{-2t} - 1$.
6. $1- e^{-t}$
7. $\frac{1}{4}\left(1 - e^{-4t}\right)$
8. $\sinh 2t - 2t$
9. $\frac{1}{4}(1 - 2t + \sin 2t - \cos 2t)$
10. $5e^{3t} - 2e^{-2t}$
11. $\frac{2}{a} - \frac{2}{a}e^{-at} - t$
12. $2e^{-at} - 1$
13. $\frac{1-\cos 2t}{4}$
14. $\frac{2(1-e^{-at})-at}{a}$
15. $\frac{e^{2t}-1-2t-2t^2}{8}$
16. $\frac{t^2-2\pi t-2+2e^{\pi t}}{2\pi^2}$
17. $\frac{t^3}{6\pi^2} - \frac{t}{\pi^4} + \frac{\sin \pi t}{\pi^5}$
18. $4e^{3t} - e^{-t}$
19. $f(t) = \frac{5}{2}\sin 2t$
20. $f(t) = \frac{2}{5}e^{t} - \frac{2}{5}e^{-4t}$
21. $f(t) = 2\cosh 2t - \frac{3}{2}\sinh 2t$
22. $f(t) = 2e^{t}\cos t + 3e^{t}\sin t$
23. $f(t) = -4e^{-2t}\cos t + 11e^{-2t}\sin t$

Chapter 18

# Transform of Derivatives and Integrals

## 18.1 Solution of Initial Value Problems

The Laplace transform is a method of solving differential equations. The crucial idea is that the Laplace transform replaces operations of calculus by operations of algebra on transforms. We will see that differentiation of $f(t)$ is replaced by multiplication of $L(s)$ by $s$. Integration of $f(t)$ is replaced by division of $L[f(t)]$ by $s$.

### 18.1.1 Laplace Transforms of Derivatives

**Theorem 1 (Differentiation of $f(t)$)**

Suppose that $f$ is continuous and $f'$ is piecewise continuous on any interval $0 \le t \le A$. suppose further that there exist constants $K$, $a$, and $M$ such that

$$|f(t)| \le \; Ke^{at} \text{ for } t \ge M.$$

Then $L[f'(t)]$ exists for $s > a$, and

$$\mathrm{L}[f'(t)] = s\mathrm{L}[f(t)] \quad - f(0). \tag{18.1}$$

**Proof** We have to show $\int_0^\infty e^{-st} f'(t)\,dt$ is convergent. For this we consider the integral $\int_0^A e^{-st} f'(t)\,dt$.

If $f'$ has points of discontinuity in the interval $0 \le t \le A$, let them be denoted by $t_1,\; t_2,\; \cdots,\; t_n$.

Then

$$\int_0^A e^{-st} f'(t)\,dt = \int_0^{t_1} e^{-st} f'(t)\,dt + \int_{t_1}^{t_2} e^{-st} f'(t)\,dt + \cdots + \int_{t_n}^{A} e^{-st} f'(t)\,dt.$$

Integrating each term on the right by parts, we obtain

$$\begin{aligned}
\int_0^A e^{-st} f'(t)\,dt \quad = \quad & \left[e^{-st} f(t)\right]_0^{t_1} + \left[e^{-st} f(t)\right]_0^{t_1} \\
& [+ \left[e^{-st} f(t)\right]_{t_1}^{t_2} + \cdots + \left[e^{-st} f(t)\right]_{t_n}^{A} \\
& [+s \left[\int_0^{t_1} e^{-st} f(t)\,dt + \int_{t_1}^{t_2} e^{-st} f(t)\,dt\right]. \\
& + \cdots + \int_{t_n}^{A} e^{-st} f(t)\,dt
\end{aligned}$$

Since $f$ is continuous, the contributions of the integrand terms at $t_1,\ t_2,\ \cdots,\ t_n$ cancel. Combining the integrals, we obtain

$$\int_0^A e^{-st} f'(t)\, dt = e^{-sA} f(A) - f(0) + s \int_0^A e^{-st} f(t)\, dt.$$

For $A \geq M$, we have $|f(A)| \leq K e^{aA}$. Hence,

$$|e^{-sA} f(A)| \leq\ K e^{-(s-a)A}$$

Hence $e^{-sA} f(A) \to 0$ as $A \to \infty$ whenever $s > a$. Therefore, for $s > a$,

$$L[f'(t)] = \int_0^\infty e^{-st} f'(t)\, dt\ = sL[f(t)] - f(0).$$

This completes the proof of the theorem.

**Corollary to Theorem 1:**

$$L\left[f''(t)\right] = s^2 L\left[f(t)\right] - sf(0) - f'(0). \tag{18.2}$$

**Proof** Applying (18.1) to$f''(t)$, we get

$$\mathrm{L}[f''(t)] = s\mathrm{L}[f'(t)]\ -\ f'(0) = s\left[s\mathrm{L}[f(t)] - f(0)\right] - f'(0).$$

That is,

$$L\left[f''(t)\right] = s^2 L\left[f(t)\right] - sf(0) - f'(0).$$

This completes the proof.

Proceeding similarly as in the Corollary above, we obtain

$$L\left[f'''(t)\right] = s^3 L\left[f(t)\right] - s^2 f(0) - sf'(0) - f''(0). \tag{18.3}$$

By induction, we obtain the following extension of the above theorem.

**Theorem 2 (Derivative of any order $n$)**

$$L\left[f^{(n)}(t)\right] = s^n L\left[f(t)\right] - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0).$$

***Example*** **1** If$f(t) = t\sin at,$ find $\mathrm{L}[f(t)]$.

*Solution*

$$f(t) = t\sin at, \text{ so } \quad f(0) = 0, \quad f'(t) = \sin at + at\cos at, \; f'(0) = 0$$

$$f''(t) = a\cos at + a\cos at - a^2 t\sin at = 2a\cos at - a^2 f(t)$$

Substituting these values in

$$L\left[f''(t)\right] = s^2 L\left[f(t)\right] - sf(0) - f'(0)$$

we obtain

$$\mathrm{L}[2a\cos at - a^2 f(t))] = s^2 \mathrm{L}[f(t)] - s\cdot 0 - 0$$

implies (using Linearity of Laplace transformation to the left hand side)

$$2a\mathrm{L}[\cos at] - \quad a^2 \mathrm{L}[f(t)] = s^2 \mathrm{L}[f(t)]$$

Bringing $\mathrm{L}[f(t)]$ to the right hand side, the above becomes

$$(s^2+a^2)\mathrm{L}[f(t)] = 2a\mathrm{L}[\cos at]$$

Noting that $\mathrm{L}[\cos at] = \frac{s}{s^2+a^2}$, the above implies

$$(s^2+a^2)\mathrm{L}[f(t)] = \frac{2as}{s^2+a^2}$$

or

$$\mathrm{L}[t\sin at] = \mathrm{L}[f(t)] = \frac{2as}{(s^2+a^2)^2}.$$

***Example* 2** If$f(t) = t\cos at$, find $\mathrm{L}[f(t)]$.

*Solution*

If we proceed as in Example 2, we obtain

$$\mathrm{L}[t\cos at] = \frac{s^2-a^2}{(s^2+a^2)^2}$$

**Table:** *Some functions $f(t)$ and their placeLaplace transforms*

| | $f(t)$ | $L[f(t)]$ |
|---|---|---|
| 1 | $t\cos at$ | $\frac{s^2-a^2}{(s^2+a^2)^2}$ |
| 2 | tsin$at$ | $\frac{2as}{(s^2+a^2)^2}$ |

***Example* 3** Find the inverse transform of $\frac{1}{(s^2+a^2)^2}$.

*Solution*

$$\begin{aligned}\frac{1}{(s^2+a^2)^2} &= \frac{1}{2a^2}\{\frac{(s^2+a^2)-(s^2-a^2)}{(s^2+a^2)^2}\} \\ &= \frac{1}{2a^2}\left\{\frac{1}{s^2+a^2}-\frac{(s^2-a^2)}{(s^2+a^2)^2}\right\}.\end{aligned}$$

Hence,

$$\begin{aligned}L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] &= \frac{1}{2a^2}L^{-1}\left[\frac{1}{s^2+a^2}\right]-\frac{1}{2a^2}\mathrm{L}^{-1}\left[\frac{s^2-a^2}{(s^2+a^2)^2}\right],\end{aligned}$$

by the Linearity Theorem of

inverse Laplace transform

$$\begin{aligned}&= \frac{1}{2a^2}\frac{1}{a}\sin at - \frac{1}{2a^2}t\cos at \\ &= \frac{1}{2a^3}\{\sin at - at\cos at\}\end{aligned}$$

***Example* 4** Find the inverse transform of $\frac{s^3}{(s^2+a^2)^2}$.

*Solution*

$$\frac{s^3}{(s^2+a^2)^2} = \frac{s[(s^2+a^2)-a^2]}{(s^2+a^2)^2} = \frac{s}{s^2+a^2} - \frac{a^2 s}{(s^2+a^2)^2}$$

$$\mathrm{L}^{-1}\left[\frac{s^3}{(s^2+a^2)^2}\right] = \mathrm{L}^{-1}\left[\frac{s}{s^2+a^2}\right] - a^2\mathrm{L}^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$$

$$= \cos at - a^2\frac{1}{2a}t\sin at = \cos at - \frac{at}{2}\sin at$$

**Exercises**

In Exercises 1 and 2, prove using differentiation theorem.

1. $L(t\cosh\, at) = \frac{s^2+a^2}{(s^2-a^2)^2}$

2. $L(t\sinh\, at) = \frac{2as}{(s^2-a^2)^2}$

### 18.1.2 Solution of Initial Value Problems

The Laplace transform is useful in solving initial value problems. Suppose we have to solve the second order linear differential equation

$$\frac{d^2y}{dt^2} + \alpha\frac{dy}{dt} + \beta y = f(t) \tag{18.4}$$

where $\alpha$ and $\beta$ are constants, subject to the initial condition $y(0) = A$, and $y'(0) = B$, where $A$ and $B$ are given constants.

Taking Laplace transform of both sides of equation (18.4) and using the given initial conditions, we obtain an algebraic equation for the determination of $\mathrm{L}\,[y(t)] = Y(s)$. The required solution $y(t)$ is then obtained by finding the inverse Laplace transform of $Y(s)$. The method can be extended to higher order differential equations.

**Notation**

Suppose we denote$\mathrm{L}\,[y(t)] = Y(s)$. Then we have

$$\begin{aligned}\mathrm{L}\left[y'(t)\right] &= sY(s) - y(0).\\ \mathrm{L}\left[y''(t)\right] &= s^2Y(s) - sy(0) - y'(0)\\ \mathrm{L}\left[y'''(t)\right] &= s^3Y(s) - s^2y(0) - sy'(0) - y''(0) \text{ and so on.}\end{aligned}$$

***Example* 5** Using Laplace transform, find the solution of

$$y'' + 4y = 4t \tag{18.5}$$

that satisfies the initial conditions $y(0) = 1$ and $y'(0) = 5$. Verify the same using another method.

*Solution*

When $L$ is applies to both sides of (18.5), we get

$$L[y''] + 4L[y] = 4L[t]. \tag{18.6}$$

Using Theorem and noting that $L(t) = 1/s^2$, (18.6) becomes

$$s^2\, L[y] - s - 5 + 4L[y] = \frac{4}{s^2}$$

or

$$(s^2 + 4)\, L[y] = s + 5 + \frac{4}{s^2},$$

so

$$\begin{aligned} L\,[y] &= \frac{s}{s^2+4} + \frac{5}{s^2+4} + \frac{4}{s^2(s^2+4)} \\ &= \frac{s}{s^2+4} + \frac{5}{s^2+4} + \frac{1}{s^2} - \frac{1}{s^2+4} \\ &= \frac{s}{s^2+4} + \frac{4}{s^2+4} + \frac{1}{s^2} \end{aligned} \tag{18.7}$$

i.e.,

$$L\,[y] = L\,[\cos 2t] + L\,[2\sin 2t] + L\,[t]$$

$$= L\,[\cos 2t + 2\sin 2t + t].$$

so

$$y = \cos 2t + 2\sin 2t + t$$

is the desired solution.

We can easily check this result, for the general solution of (18.5) is seen by inspection to be

$$y = c_1 \cos\, 2t + c_2 \sin\, 2t + t,$$

and the initial conditions imply at once that $c_1 = 1$ and $c_2 = 2$.

***Example* 6** Solve the following initial value problem:

$$y'' + y = \sin 2t, \tag{18.8}$$

with

$$y(0) = 2, \quad y'(0) = 1 \tag{18.9}$$

*Solution*

Taking the Laplace transform of the differential equation, we have

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{2}{s^2+4}.$$

Substituting for $y(0)$ and $y'(0)$from the initial conditions and solving for $Y(s)$, we obtain

$$Y(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2+1)(s^2+4)} \tag{18.10}$$

Using partial fractions, we can write $Y(s)$in the form

$$Y(s) = \frac{as+b}{s^2+1} + \frac{cs+d}{s^2+4} = \frac{(as+b)(s^2+4)+(cs+d)(s^2+1)}{(s^2+1)(s^2+4)} \tag{18.11}$$

By expanding the numerator on the right side of Eq.(18.11) and equating it to the numerator in Eq.(18.10), we find that

$$2s^3 + s^2 + 8s + 6 = (a+c)s^3 + b(+d)s^2 + (4a+c)s + (4b+d)$$

for all $s$. Then, comparing coefficients of like powers of $s$, we have

$$a + c = 2, \quad b + d = 1,$$

$$4a + c = 8, \quad 4b + d = 6.$$

Hence,

$a = 2,\ c = 0,\ b = \frac{5}{3}$ and $d = -\frac{2}{3}$,

from which we obtain

$$Y(s) = \frac{2s}{s^2+1} + \frac{5/3}{s^2+1} - \frac{2/3}{s^2+4} \tag{18.12}$$

Hence

$$y(t) = L^{-1}[Y(s)] = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin 2t. \tag{18.13}$$

***Example 7*** A small body of mass $m = 2$ is attached at the lower end of an elastic spring whose upper end is fixed, the spring modulus being $k = 10$. Let $y(t)$ be the displacement of the body from the position of static equilibrium. Determine the free vibrations of the body, starting from the initial position $y(0) = 2$ with the initial velocity $y'(0) = 4$, assume that there is damping proportional to the velocity, the damping constant being $c = 4$.

*Solution*

The differential equation governing the motion of the damped mechanical system is

$$my'' + cy' + ky = 0.$$

Here using the given values, the differential equation is

$$2y'' + 4y' + 10y = 0$$

or

$$y'' + 2y' + 5y = 0.$$

Also the initial conditions are given by $y(0) = 2$ and $y'(0) = 4$.

i.e., we have solve the initial value problem

$y'' + 2y' + 5y = 0$; $y(0) = 2$ and $y'(0) = 4$.

Taking Laplace transform of both sides of the above differential equation, we get

$$\mathrm{L}(y'') + 2\mathrm{L}(y') + 5\mathrm{L}(y) = 0.$$

i.e., $s^2Y(s) - sy(0) - y'(0) + 2[sY(s) - y(0)] + 5Y(s) = 0$,

where $\mathrm{L}\,[y(t)] = Y(s)$

i.e., $s^2Y(s) - s\cdot 2 - 4 + 2[sY(s) - 2] + 5Y(s) = 0$

i.e., $(s^2 + 2s + 5)Y(s) - 2s = 8$ or $Y(s) = \frac{2s+8}{s^2+2s+5}$.

Now

$$Y(s) = \frac{2s}{s^2 + 2s + 5} = \frac{2s}{(s+1)^2 + 2^2} = 2\cdot\frac{s+1}{(s+1)^2 + 2^2} + \frac{3\cdot 2}{(s+1)^2 + 2^2}$$

$$y(t) = \mathrm{L}^{-1}[Y(s)] = 2\mathrm{L}^{-1}\left[\frac{s+1}{(s+1)^2 + 2^2}\right] + 3\mathrm{L}^{-1}\left[\frac{2}{(s+1)^2 + 2^2}\right]$$

$$= 2e^{-t}\cos 2t + 3e^{-t}\sin 2t = e^{-t}[2\cos 2t + 3\sin 2t]$$

***Example* 8** Find the solution of the initial value problem

$$y^{(4)} - y = 0 \tag{18.14}$$

$$y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0, \quad y'''(0) = 0 \tag{18.15}$$

*Solution*

The Laplace transform of the differential equation (18.14) is

$$s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) - Y(s) = 0.$$

Then, using the initial conditions (18.15) and solving for $Y(s)$, we have

$$Y(s) = \frac{s^2}{s^4 - 1} \tag{18.16}$$

A partial fraction expansion of $Y(s)$ is

$$Y(s) = \frac{as+b}{s^2-1} + \frac{cs+d}{s^2+1} \tag{18.17}$$

and it follows that

$$(as+b)(s^2+1) + (cs+d)(s^2-1) = s^2 \tag{18.18}$$

for all $s$. By setting $s = 1$ and $s = -1$, respectively in Eq.(18.17), we obtain the pair of equations

$$2(a+b) = 1, \quad 2(-a+b) = 1,$$

and therefore $a = 0$ and $b = \frac{1}{2}$. If we set $s = 0$ in Eq.(18.18) then $b - d = 0$, so $d = \frac{1}{2}$. Finally, equating the coefficients of the cubic terms on each side of Eq.(18.18), we find that $a + c = 0$, so $c = 0$. Thus,

$$Y(s) = \frac{1/2}{s^2-1} + \frac{1/2}{s^2+1}. \tag{18.19}$$

Hence

$$y(t) = L^{-1}[Y(s)] = \frac{\sinh t + \sin t}{2}. \tag{18.20}$$

*More initial value problems will be discussed in the coming chapters related to Laplace transforms.*

**Exercises**

In Exercises 1-9, solve the following initial value problems by the Laplace transform.

1. $y' + 3y = 10\sin t,\ \ y(0) = 0$

2. $y' + 0.2y = 0.01t,\ \ y(0) = -0.25$

3. $y'' + ay' - 2a^2y = 0y(0) = 6, \quad y'(0) = 0$

4. $y'' - 4y' + 3y = 6t - 8, y(0) = 0, \quad y'(0) = 0$

5. $y'' + 2y' - 3y = 6e^{-2t}, y(0) = 2, \quad y'(0) = -14$

6. $y' - 5y = 1.5e^{-4t},\ \ y(0) = 1$

7. $y'' - y' - 2y = 0,\ \ y(0) = 8, \quad y'(0) = 7$

8. $y'' + y = 2\cos ty(0) = 3, \quad y'(0) = 4$

9. $y'' + 0.04y = 0.0t^2y(0) = -25, \quad y'(0) = 0$

10. $y'' - y' - 6y = 0; \quad y(0) = 1, \ \ y'(0) = -1$

11. $y'' + 3y' + 2y = 0; \quad y(0) = 1, \quad y'(0) = 0$

12. $y'' - 2y' + 4y = 0; \quad y(0) = 2, \quad y'(0) = 0$

13. $y^{(4)} - 4y''' + 6y'' - 4y' + y = 0$; $y(0) = 0$, $y'(0) = 1$, $y''(0) = 0$, $y'''(0) = 1$

14. $y^{(4)} - y = 0; \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 1, \quad y'''(0) = 0$

15. $y'' + 2y' + y = 4e^{-t}; \quad y(0) = 2, \ \ y'(0) = -1$

16. $y'' - 2y' + 2y = \cos t; \quad y(0) = 1, \quad y'(0) = 0$

***Answers***

1. $y = e^{-3t} - \cos t + 3\sin t$

2. $y = 0.05t - 0.25$

3. $y = 2e^{-2at} + 4e^{at}$

4. $y = 2t + e^{t} - e^{3t}$

5. $y = -2e^{-2t} + \frac{11}{2}e^{-3t} - \frac{3}{2}e^{t}$

10. $y = \frac{1}{5}(e^{3t} + 4e^{-2t})$

11. $.y = 2e^{-t} - e^{-2t}$

12. $y = 2e^{t}\cos\sqrt{3}\,t - (2/\sqrt{3})e^{t}\sin\sqrt{3}\,t$

13. $y = te^{t} - t^{2}e^{t} + \frac{2}{3}t^{3}e^{t}$

14. $y = \cosh t$

15. $y = 2e^{-t} + te^{-t} + 2t^{2}e^{-t}$

16. $y = \frac{1}{5}(\cos t - 2\sin t + 4e^{t}\cos t - 2e^{t}\sin t)$

## 18.2 Laplace Transform of the Integral of a Function

**Theorem 3 (Integration of $f(t)$ )**

$$\mathrm{L}\left[\int_0^t f(u)du\right] = \frac{1}{s}\mathrm{L}[f(t)]; \quad (s > 0,\ \ s > \gamma).$$

**Proof** Take

$$g(t) = \int_0^t f(u)du.$$

Then

$$f(t) = g'(t).$$

Hence, by Theorem 1,

$$\mathrm{L}[f(t)] = \mathrm{L}[g'(t)] = s\mathrm{L}[g(t)] - g(0) \tag{18.21}$$

Here, $g(0) = \int_0^0 f(u)du = 0$, (18.21) becomes

$$\mathrm{L}[f(t)] = s\mathrm{L}[g(t)],$$

which implies that

$$\mathrm{L}[g(t)] = \frac{1}{s}\mathrm{L}[f(t)]$$

or

$$\mathrm{L}\left[\int_0^t f(u)du\right] = \frac{1}{s}\mathrm{L}[f(t)].$$

**Corollary to Theorem 3**

$$\int_0^t f(u)du = \mathrm{L}^{-1}\left[\frac{1}{s}F(s)\right] \tag{18.22}$$

**Proof** This is obtained if we put $\mathrm{L}[f(t)] = F(s)$in the above result

***Example* 9** If $\mathrm{L}[f(t)] = \frac{1}{s^2(s^2+a^2)}$, find *f(t)*.

*Solution*

We do the work in two steps:

**Step 1:** Evaluation of $\mathrm{L}^{-1}\left[\frac{1}{s}\cdot\frac{1}{s^2+a^2}\right]$.

By taking$F(s)=\frac{1}{s^2+a^2}$, and noting that

$$f(t)=\mathrm{L}^{-1}[F(s)]=\mathrm{L}^{-1}\left[\frac{1}{s^2+a^2}\right]=\frac{1}{a}\sin at,$$

(18.22) gives

$$\begin{aligned}
\mathrm{L}^{-1}\left[\frac{1}{s}\cdot\frac{1}{s^2+a^2}\right] &= \mathrm{L}^{-1}\left[\frac{1}{s}F(s)\right]=\int_0^t f(u)du \\
&= \int_0^t \frac{1}{a}\sin au\,du \\
&= \frac{1}{a}\left[\frac{-\cos au}{a}\right]_0^t \\
&= \frac{1}{a^2}\left[1-\cos at\right]
\end{aligned}$$

**Step 2:** Evaluation of $\mathrm{L}^{-1}\left[\frac{1}{s^2}\cdot\frac{1}{s^2+a^2}\right]$.

Now taking

$$F(s)=\frac{1}{s}\cdot\frac{1}{s^2+a^2},$$

and noting from Step 1 that

$$f(t)=\mathrm{L}^{-1}[F(s)]=\mathrm{L}^{-1}\left[\frac{1}{s}\cdot\frac{1}{s^2+a^2}\right]=\frac{1}{a^2}\left[1-\cos at\right],$$

(18.22) gives

$$\begin{aligned}
L^{-1}\left[\frac{1}{s^2}\cdot\frac{1}{s^2+a^2}\right] &= L^{-1}\left[\frac{1}{s}.\left(\frac{1}{s}\cdot\frac{1}{s^2+a^2}\right)\right]\\
&= L^{-1}\left[\frac{1}{s}F(s)\right] = \int_0^t f(u)du\\
&= \int_0^t \frac{1}{a^2}\left[1-\cos au\right]du
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{a^2}\left[u-\frac{\sin au}{a}\right]_0^t\\
&= \frac{1}{a^2}\left[t-\frac{\sin at}{a}\right]
\end{aligned}$$

***Example* 10** If $L[f(t)] = \frac{s+1}{s^2(s^2+1)}$, find *f(t)*.

*Solution*

We do the work in two steps:

**Step 1:** Evaluation of $L^{-1}\left[\frac{1}{s}\cdot\frac{s+1}{s^2+1}\right]$: Take $F(s) = \frac{s+1}{s^2+1}$, then

$$\begin{aligned}
f(t) &= L^{-1}[F(s)] = L^{-1}\left[\frac{s+1}{s^2+1}\right]\\
&= L^{-1}\left[\frac{s}{s^2+1}\right] + L^{-1}\left[\frac{1}{s^2+1}\right] = \cos t + \sin t.
\end{aligned}$$

Hence by (18.22),

$$\begin{aligned}
\mathrm{L}^{-1}\left[\frac{1}{s}\cdot\frac{s+1}{s^2+1}\right] &= \mathrm{L}^{-1}\left[\frac{1}{s}F(s)\right] \\
&= \int_0^t f(u)du = \int_0^t (\cos u + \sin u)du \\
&= [\sin u - \cos u]_0^t = \sin t - \cos t + 1.
\end{aligned}$$

**Step 2:** Evaluation of $\mathrm{L}^{-1}\left[\frac{1}{s^2}\cdot\frac{s+1}{s^2+a^2}\right]$.

Now take

$$F(s) = \frac{1}{s}\cdot\frac{s+1}{s^2+1},$$

then by Step 1

$$f(t) = L^{-1}[F(s)] = \sin t - \cos t + 1$$

and by (18.22),

$$\begin{aligned}
\mathrm{L}^{-1}\left[\frac{1}{s^2}\cdot\frac{s+1}{s^2+1}\right] &= \mathrm{L}^{-1}\left[\frac{1}{s}.\left(\frac{1}{s}\frac{1}{s^2+a^2}\right)\right] \\
&= \mathrm{L}^{-1}\left[\frac{1}{s}F(s)\right] = \int_0^t f(u)du \\
&= \int_0^t (\sin u - \cos u + 1)du \\
&= 1 - \cos t - \sin t + t.
\end{aligned}$$

**Exercises**

Find inverse transforms by integration.

1. $\frac{1}{s^2+4s}$
2. $\frac{1}{s(s^2+\omega^2)}$
3. $\frac{1}{s^3-s}$
4. $\frac{9}{s^2}\left(\frac{s+1}{s^2+9}\right)$
5. $\frac{4}{s^3+2s^2}$
6. $\frac{1}{s^5+s^3}$
7. $\frac{1}{s^2}\left(\frac{s-1}{s+1}\right)$
8. $\frac{\pi^5}{s^4(s^2+\pi^2)}$

***Answers***

1. $\frac{1}{4}(1-e^{-4t})$
2. $(1-\cos\omega t)/\omega^2$
3. $\cosh t - 1$
4. $1+t-\cos 3t - \frac{1}{3}\sin 3t$

Chapter 19

# Unit Step Function and Impulse Function

In this chapter we define two important functions, the unit step function and Dirac's delta function. We also state and prove shifting theorems.

**Unit Step Function**

The **unit step function** or **Heaviside function,** denoted by $u_c$, is defined by

$$u_c(t) = \begin{cases} 0, & \text{if} \quad t < c \\ 0, & \text{if} \quad t \geq c \end{cases} \qquad (c \geq 0 \text{ is fixed})$$

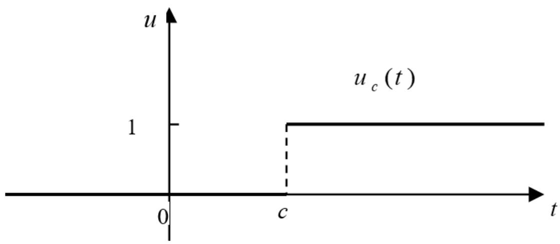


Figure 19.1:

**Notation** The unit step function $u_c(t)$ is sometimes denoted by $u(t-c)$.

***Example* 1** Given

$$h(t) = u_\pi(t) - u_{2\pi}(t), \quad t \geq 0.$$

Give the explicit form of $h(t)$.

*Solution*

Using the definition of $u_c(t)$, we have

$$h(t) = \begin{cases} 0 - 0 = 0, & 0 \leq t < \pi, \\ 1 - 0 = 1, & \pi \leq t < 2\pi, \\ 1 - 1 = 0, & 2\pi \leq t < -\infty, \end{cases}$$

***Example* 2** Express the function

$$f(t) = \begin{cases} 2, & 0 \le t < 4, \\ 5, & 4 \le t < 7, \\ -1, & 7 \le t < 9, \\ 1, & t \ge 9, \end{cases} \tag{19.1}$$

in terms of unit step functions.

*Solution*

We start with the function $f_1(t) = 2$, which agrees with $f(t)$ on $[0, 4)$. To produce the jump of three units at $t = 4$, we add $3u_4(t)$ obtaining

$$f_2(t) = 2 + 3u_4(t),$$

which agrees with $f(t)$ on $[0, 7)$. The negative jump of six units at $t = 7$ corresponds to adding $-6u_7(t)$, which gives

$$f_3(t) = 2 + 3u_4(t) - 6u_7(t).$$

Finally, we must add $2u_9(t)$ to match the jump of two units at $t = 9$. Thus we obtain

$$f(t) = 2 + 3u_4(t) - 6u_7(t) + 2u_9(t). \tag{19.2}$$

### The Laplace Transform of $u_c$

The Laplace transform of $u_c$is determined as follows:

$$\mathrm{L}[u_c(t)] = \int_0^\infty e^{-st}u_c(t)dt = \int_0^c e^{-st}0dt + \int_c^\infty e^{-st}dt$$

$$= \left[\frac{e^{-st}}{s}\right]_c^\infty = \frac{e^{-cs}}{s} \quad (s > 0).$$

For a given function $f$ determined for $t \geq 0$, we will often want to consider the related function $g$ defined by

$$y = g(t) = \begin{cases} 0 & \text{when } \ t < c, \\ f(t-c) & \text{when } \ t \geq c, \end{cases}$$

which represents a translation of $f$ a distance $c$ in the positive $t$ direction. In terms of the unit step function we can write $g(t)$ in the convenient form

$$g(t) = u_c(t)f(t-c).$$

The unit step function is particularly important in transform use because of the relation (in the following theorem) between the transform of $f(t)$ and that of its translation $u_c(t)f(t-c)$.

**Theorem 1 (Shifting on the $t$-axis)**

If $F(s) = \mathrm{L}[f(t)]$ exists for $s > a \geq 0$, and if $c$ is a positive constant, then

$$\mathrm{L}\ [u_c(t)f(t-c)] = e^{-cs}\mathrm{L}[f(t)] = e^{-cs}F(s), \quad s > a. \qquad (19.3)$$

**Proof**

$$\begin{aligned}
\mathrm{L}[u_c(t)f(t-c)] &= \int_0^{\infty} e^{-st}u_c(t)f(t-c)dt \\
&= \int_0^{c} e^{-st}u_c(t)f(t-c)dt + \int_c^{\infty} e^{-st}u_c(t)f(t-c)dt \\
&= \int_c^{\infty} e^{-st}f(t-c)dt \\
&= \int_0^{\infty} e^{-s(c+u)}f(u)du, \text{ by putting } t-c=u, \\
&\quad \text{and noting } u=0, \text{ when } t=c \\
&= e^{-cs}\int_0^{\infty} e^{-su}f(u)du \\
&= e^{-cs}\mathrm{L}[f(u)] \;=\; e^{-cs}F(s)
\end{aligned}$$

**Corollary 1** If $f(t) = \mathrm{L}^{-1}[F(s)]$, then

$$\mathrm{L}^{-1}\left[e^{-cs}F(s)\right] = u_c(t)\,f\,(t\;-\;c) \tag{19.4}$$

**Proof** (19.4) is obtained if we apply inverse Laplace transformation on both sides of (19.3).

***Example* 3** Find $\mathrm{L}[f(t)]$, where

$$f(t) = \begin{cases} \sin\,t, & 0 \le t < \frac{\pi}{4} \\ \sin\,t\,+\cos\,(t-\frac{\pi}{4}), & t \ge \frac{\pi}{4} \end{cases}$$

*Solution*

Note that

$$f(t) = \sin\,t + g(t),$$

where

$$g(t) = \begin{cases} 0, & t < \frac{\pi}{4} \\ \\ \cos\left(t - \frac{\pi}{4}\right), & t \geq \frac{\pi}{4} \end{cases}$$

$$= u_{\frac{\pi}{4}}(t) \cos\left(t - \tfrac{\pi}{4}\right).$$

Hence, by the linearity of Laplace transform,

$$\begin{aligned}
\mathrm{L}[f(t)] &= \mathrm{L}[\sin t] + \mathrm{L}[g(t)] \\
&= \mathrm{L}[\sin t] + \mathrm{L}[u_{\frac{\pi}{4}}(t)\cos(t - \tfrac{\pi}{4})] \\
&= \mathrm{L}[\sin t] + e^{-\frac{\pi s}{4}}\mathrm{L}[\cos t], \text{ using Theorem 1} \\
&= \frac{1}{s^2+1} + e^{-\frac{\pi s}{4}} \cdot \frac{\mathrm{s}}{s^2+1} \\
&= \frac{1 + se^{-\frac{\pi s}{4}}}{s^2+1}.
\end{aligned}$$

***Example* 4** Find the inverse Laplace transform of $\frac{1-e^{-2s}}{s^2}$.

*Solution*

$$\begin{aligned}
\mathrm{L}^{-1}\left[\frac{1-e^{-2s}}{s^2}\right] &= \mathrm{L}^{-1}\left[\frac{1}{s^2}\right] - \mathrm{L}^{-1}\left[\frac{e^{-2s}}{s^2}\right] \\
&= t - u_2(t)\ (t-2), \text{ since } \mathrm{L}^{-1}\left[\frac{1}{s^2}\right] = t \\
&\quad \text{and using Corollary to Theorem 1} \\
&= \begin{cases} t, & 0 \leq t < 2 \\ \\ 2, & t \geq 2 \end{cases}
\end{aligned}$$

**Corollary 2**

$$\mathrm{L}[u_c(t)] = \frac{e^{-cs}}{s}, \quad (s > 0). \tag{19.5}$$

**Proof**

With $f(t-c) = 1$ for all $t$, Theorem 1 gives

$$\mathrm{L}[u_c(t)\,] = e^{-cs} \cdot \frac{1}{s},$$

since $\mathrm{L}[1] = \frac{1}{s}$

***Example* 5** Find the Laplace transform of the function

$$g(t) = \begin{cases} 1, & \text{if} \quad 0<t<\pi \\ 0, & \text{if} \quad \pi < t < 2\pi \\ \sin t, & \text{if} \quad t>2\pi \end{cases}$$

*Solution*

We start with the function

$$f_1(t) = 1,$$

which agrees with $g(t)$ on $0 < t < \pi$. The negative jump of one units at $t = \pi$ corresponds to adding $-u_\pi(t)$, which gives

$$f_2(t) = 1 - u_\pi(t).$$

$f_2$ agrees with $g$ on $0 < t < 2\pi$. At $t = 2\pi$ we want $\sin t$ to come on, so we have to add $u_{2\pi}(t)$ sin $t$, which gives

$$g(t) = 1 - u_\pi(t) + u_{2\pi}(t) \sin t.$$

Here $\sin t = \sin(t - 2\pi)$, because of the periodicity. Hence the above can be written as

$$g(t) = 1 - u_\pi(t) + u_{2\pi}(t) \sin(t - 2\pi).$$

Hence,

$$\mathrm{L}[g(t)] = \mathrm{L}[1] - \mathrm{L}[u_\pi(t)] + \mathrm{L}\left[u_{2\pi}(t)\sin(t - 2\pi)\right]$$

Now

$$\mathrm{L}[1] = \frac{1}{s},$$

and ( by Corollary 2)

$$\mathrm{L}[u_\pi(t)] = \frac{e^{-\pi s}}{s}.$$

Taking $f(t) = \sin t$, and noting L $[\sin\ t] = F(s) = \frac{1}{s^2+1}$, Theorem 1 gives

$$\mathrm{L}[u_{2\pi}(t)\ \sin(t-2\pi) = \mathrm{L}[u_{2\pi}(t)f(t-2\pi)] \ = \ e^{-2\pi s}F(s) = e^{-2\pi s}\frac{1}{s^2+1}.$$

Hence

$$\mathrm{L}[g(t)] = \frac{1}{s} - \frac{e^{-\pi s}}{s} + \frac{e^{-2\pi s}}{s^2+1}.$$

***Example* 6** Find the inverse transform of $\frac{e^{-3s}}{s^3}$.

*Solution*

Taking $F(s)=\frac{1}{s^3}$, we have $f(t)=\mathrm{L}^{-1}\ \left[\frac{1}{s^3}\right]=\frac{t^2}{2}$, and hence

$$\begin{aligned}\mathrm{L}^{-1}\left[\frac{e^{-3s}}{s^3}\right] &= u_3(t)\frac{(t-3)^2}{2},\ \text{by Corollary 1}\\ &= \begin{cases}\frac{(t-3)^2}{2} & \text{if}\ \ t\geq 3\\ 0 & \text{if}\ \ t<3\end{cases}\end{aligned}$$

***Example* 7** Solve the initial value problem $\frac{d^2y}{dt^2}+2y=r(t),\ \ y(0)=0,\ \ y'(0)=0$,

where

$$r(t)=\begin{cases}1 \ \text{if}\ \ t\geq 1\\ 0\ \text{if}\ \ t<1\end{cases}$$

*Solution*

Here note that $r(t)$ is the step function $u_1(t)$. Now taking the Laplace transform of both sides of the given differential equation, we get

$$\mathrm{L}(y'')+2\mathrm{L}(y)=\mathrm{L}\left(u_1(t)\right).$$

i.e., using (19.5),

$$s^2Y(s)-sy(0)-y'(0)+2Y(s)=\frac{e^{-s}}{s},$$

where $\mathrm{L}\left[y(t)\right]=Y(s)$

i.e.,

$$(s^2+2)Y(s)=\frac{e^{-s}}{s}$$

or

$$Y(s) = \frac{e^{-s}}{s(s^2+2)} = \frac{e^{-s}}{2}\left(\frac{1}{s} - \frac{s}{s^2+2}\right).$$

Now

$L^{-1}\left[e^{-s}\frac{1}{s}\right] = 1 \cdot u(t-1)$, using (19.3)

and

$L^{-1}\left[e^{-s}\frac{s}{s^2+(\sqrt{2})^2}\right] = \cos(\sqrt{2}(t-1))\, u(t-1)$, using (19.3)

Hence

$$y(t) = L^{-1}[Y(s)]$$
$$= \frac{1}{2}\left[1 - \cos\sqrt{2}(t-1)\right] u(t-1)$$

**Exercises**

In Exercises 1-13, solve the differential equation using Laplace transform.

1. $y'' - 2y' - 8y = 4, \quad y(0) = 0, \quad y'(0) = 1.$

2. $4y'' + y = 0, \quad y(0) = 0, \quad y'(0) = 2.$

3. $y'' + 4y' + 5y = 0, \quad y(0) = 1, \quad y'(0) = 2.$

4. $y'' + 2y' + 2y = 0, \quad y(0) = 0, \quad y'(0) = 1.$

5. $y'' + 3y' + 2y = e^{-t}, \quad y(0) = 0, \quad y'(0) = 0.$

6. $y'' + 4y = \sin 2t, \quad y(0) = 0, \quad y'(0) = 0.$

7. $y'' + 4y' + 3y = 0, \quad y(0) = 3, \quad y'(0) = 1.$

8. $y'' + 4y' + 3y = e^{-t}, \quad y(0) = \quad y'(0) = 1.$

9. $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t; \quad x = 2, \quad \frac{dx}{dt} = -1 \quad \text{at} \quad t = 0.$

10. $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 5y = e^{-2t}; \quad y = 0, \quad \frac{dy}{dt} = 1 \quad \text{when } t = 0$

11. $y'' + 2y' - 3y = 10\sinh 2t, \quad y(0) = 0, \quad y'(0) = 4.$

12. $y'' + 6y' + 9y = \sin x, \quad y(0) = 1, \quad y'(0) = 0.$

13. $(D^2 + 4D + 13)y = e^{-t}\sin t, \quad y(0) = y'(0) = 1.$

In Exercises 14-17, find the inverse Laplace transform of the given function.

14. $\frac{se^{-\pi s}}{s^2+4}$

15. $\frac{e^{-\pi s}}{s^2+2s+2}$

16. $e^{-2s}\frac{s}{s^2+5^2}17.e^{-5s}\frac{s}{(s+3)^2+1}$

***Answers***

1. $y = \frac{1}{3}e^{4t} + \frac{1}{6}e^{-2t} - \frac{1}{2}$

2. $y = 4\sin\frac{t}{2}$

3. $y = e^{-2t}(\cos t + 4\sin t)$

4. $y = e^{-t}\sin t$

5. $y = te^{-t} - e^{-t} + e^{-2t}$

6. $y = \frac{1}{8}(\sin 2t - 2t\cos 2t)$

7. $y = 5e^{-t} - 2e^{-3t}$

8. $y = \frac{1}{4}(7e^{-t} + 2te^{-1} - 3e^{-3t})$

9. $e^t(2 - 3t + 3t^2)10.\frac{1}{6}(3e^{-t} - 2e^{-2t} - e^{-5t})$

10. $y = e^{2t} + \frac{5}{3}e^{-2t} - 2e^{-3t} - \frac{2}{3}e^t$

14. $\cos 2(t - \pi)$ if $t > \pi$ and $= 0$ otherwise

16. $\cos 5(t - 2)$ if $t \geq 2$ and $= 0$ otherwise

17. $[\cos(t - 5) - 3\sin(t - 5)]\, e^{-3(t-5)}$ if $t \geq 5$ and $= 0$ otherwise

## 19.1 Shifting on the s-axis [First Shifting Theorem]

**Theorem 4 (Shifting on the s-axis)**

If $F(s) = L[f(t)]$, where $s > a \geq 0$ and if $c$ is a constant, then

$$\mathrm{L}[e^{ct}f(t)] = F(s - c), \qquad s > a + c. \tag{19.6}$$

Thus, replacing $s$ by $s - c$ in the transform ("shifting on the $s$-axis") corresponds to multiplying the original function by $e^{ct}$. (Ref. Fig.)

**Proof** By definition

$$F(s) = \mathrm{L}[f(t)] = \int_0^\infty e^{-st}f(t)dt$$

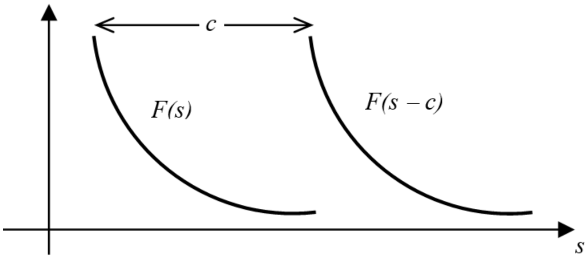


Figure 19.2:

Hence, with $s-c$ in place of $s$, we have

$$F(s-c) = \int_0^\infty e^{-(s-c)t} f(t)dt = \int_0^\infty e^{-st}\left[e^{ct}f(t)\right]dt = \mathrm{L}[e^{ct}f(t)].$$

This completes the proof.

The following result is an immediate consequence of the theorem.

**Corollary 1**

$$\mathrm{L}[e^{-ct}f(t)] = F(s+c)$$

***Example* 8** Find $L\left(e^{ct}t^n\right)$ $(n = 1,\ 2,\ \dots)$

*Solution*

Here take $f(t) = t^n$, so $F(s) = L[f(t)] = L[t^n] = \frac{n!}{s^{n+1}}$

Hence

$$L\left(e^{ct}t^n\right) = \mathrm{L}[e^{ct}f(t)] = F(s-c) = \frac{n!}{(s-c)^{n+1}}.$$

***Example* 9** Find $L\left(e^{ct}\cos bt\right)$

*Solution*

If we take $f(t) = \cos bt$, then

$$F(s) = L[f(t)] = L[\cos bt] = \frac{s}{s^2 + b^2}.$$

Hence

$$\begin{aligned} L\left(e^{ct}\cos bt\right) &= \mathrm{L}[e^{ct} f(t)] = F(s-c)] \\ &= \frac{s-c}{(s-c)^2 + b^2}. \end{aligned}$$

***Example* 10** Show that $L\left(e^{ct}\sin bt\right) = \frac{b}{(s-c)^2+b^2}$.
*Solution* is left as an exercise.

## Exercises

In Exercises 1-6, find the Laplace transforms of the given functions.

1. $2te^t$
2. $t^2 e^{-2t}$
3. $e^{-2t}\cos t$
4. $e^{-at}[A\cos bt + B\sin bt]$
5. $e^t\left(\cosh 2t + \frac{1}{2}\sinh 2t\right)$
6. $e^{-t}\left(2\cos 3t - \sin 3t\right)$

### *Answers*

1. $\frac{2}{(s+2)^3}$
2. $\frac{2}{(s+2)^3}$
3. $\frac{s+2}{(s+2)^2+1}$
4. $\frac{A(s+a)+Bb}{(s+a)^2+b^2}$
5. $\frac{s}{(s-1)^2-4}$
6. $\frac{2(s+2)-1}{(s+2)^2+9}$

**Table:** *Some functions $f(t)$ and their Laplace transforms*

| | $f(t)$ | $\mathrm{L}[f(t)]$ |
|---|---|---|
| 1 | $e^{ct}t^n, \;\; n = 1,\; 2,\; \ldots$ | $\dfrac{n!}{(s-c)^{n+1}}$ |
| 2 | $e^{ct}\cos bt$ | $\dfrac{s-c}{(s-c)^2+b^2}$ |
| 3 | $e^{ct}\sin bt$ | $\dfrac{b}{(s-c)^2+b^2}$ |

**Inverse Laplace Transform using Theorem 4**

The following result is an immediate consequence of Theorem 4.

**Corollary 2 (Inverse Laplace transform using Theorem 4)**

If

$$\mathrm{L}[f(t)] = F(s),$$

then

$$\mathrm{L}^{-1}\left[F(s-c)\right] = e^{ct}f(t) = e^{ct}\mathrm{L}^{-1}\left[F(s)\right]. \tag{19.7}$$

***Example* 11** Find $L^{-1}\left[\frac{1}{(s-1)^4}\right]$.

*Solution*

Take$F\left(s\right)=\frac{1}{s^4}$, then $f\left(t\right)=L^{-1}\left[F\left(s\right)\right]=L^{-1}\left[\frac{1}{s^4}\right]=\frac{t^3}{3\,!}$
Hence by (19.7) above,

$$L^{-1}\left[\frac{1}{\left(s-1\right)^4}\right]=\mathrm{L}^{-1}\left[F\left(s-1\right)\right]=e^{1t}f\left(t\right)=e^t\frac{t^3}{3\,!}=\frac{t^3e^t}{6}.$$

***Example* 12** Find the inverse transform of $\frac{s+2}{(s+2)^2+1}$.

*Solution*

Take $F(s)=\frac{s}{s^2+1}$, then $f(t)=L^{-1}[F(s)]=\cos t$
Here $F(s-c)=F(s-(-2))=\frac{s+2}{(s+2)^2+1}$.
Hence

$$\mathrm{L}^{-1}\left[F(s-c)\right]=e^{ct}f(t)=e^{ct}\mathrm{L}^{-1}[F(s)]=e^{ct}\mathrm{L}^{-1}\left[\frac{s}{s^2+1}\right]=e^{-2t}\cos t.$$

***Example* 13** Find the inverse transform of $\frac{s}{(s-1)^2-4}$.

*Solution*

$$\frac{s}{(s-1)^2-4}=\frac{s-1+1}{(s-1)^2-2^2}=\frac{s-1}{(s-1)^2-2^2}+\frac{1}{(s-1)^2-2^2}$$

Taking$F(s-c)=F(s-1)=\frac{s-1}{(s-1)^2-2^2}$, we have

$$L^{-1}\left(\frac{s-1}{(s-1)^2-2^2}\right)=e^t\cosh 2t$$

Similarly taking$F\left(s-a\right)=F\left(s-1\right)=\frac{1}{(s-1)^2-2^2}$, we have

$$L^{-1}\left(\frac{1}{(s-1)^2-2^2}\right)=e^t\frac{\sinh 2t}{2}$$

$$\therefore L^{-1}\left(\frac{s}{(s-1)^2-4}\right) = e^t\left(\cosh 2t + \frac{\sinh 2t}{2}\right)$$

***Example* 14** Find the inverse transform of

$$\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$$

*Solution*

To apply the methods of partial fractions, we write

$$\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} = \frac{As+B}{s^2+2s+2} + \frac{Cs+D}{s^2+2s+5} \qquad (19.8)$$

and obtain

$$s^2+2s+3 = (As+B)(s^2+2s+5) + (Cs+D)(9s^2+2s+2)$$

Equating like powers of $s$, we obtain

$$A = C = 0, \;\; B = \frac{1}{3}, \;\; D = \frac{2}{3}.$$

Now (19.8) can be written as

$$\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} = \frac{1/3}{(s+1)^2+1^2} + \frac{2/3}{(s+1)^2+2^2},$$

and so

$$\mathrm{L}^{-1}\left[\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right]$$

$$\begin{aligned} &= \frac{1}{3}\mathrm{L}^{-1}\left[\frac{1}{(s+1)^2+1^2}\right] \\ &\quad +\frac{2}{3}\mathrm{L}^{-1}\left[\frac{1}{(s+1)^2+2^2}\right] \end{aligned}$$

$$\begin{aligned} &= \frac{1}{3}e^{-t}\sin t + \frac{1}{3}e^{-t}\sin 2t \\ &= \frac{1}{3}e^{-t}\left[\sin t + \sin 2t\right]. \end{aligned}$$

***Example* 15** Find $L^{-1}\left[\frac{s}{(s+3)^2+1}\right]$.

*Solution*

**Step 1:**

$$\frac{s}{(s+3)^2+1} = \frac{s+3-3}{(s+3)^2+1} = \frac{s+3}{(s+3)^2+1} - \frac{3}{(s+3)^2+1} \quad (19.9)$$

**Step 2:** Evaluation of $L^{-1}\left[\frac{s+3}{(s+3)^2+1}\right]$.

Taking$F(s) = \frac{s}{s^2+1}$, we get $f(t) = L^{-1}\left[F(s)\right] = L^{-1}\left[\frac{s}{s^2+1}\right] = \cos t$

Hence by (19.9) above,

$$L^{-1}\left[\frac{s+3}{(s+3)^2+1}\right] = \mathrm{L}^{-1}\left[F(s-(-3))\right] = e^{-3t}f(t) = e^{-3t}\cos t$$

**Step 3:** Proceeding similarly,

$$L^{-1}\left[\frac{3}{(s+3)^2+1}\right] = 3e^{-3t}\sin t.$$

Using (19.9) and Linearity Theorem for Inverse transform, we obtain

$$\therefore L^{-1}\left[\frac{s}{(s+3)^2+1}\right]=e^{-3t}\cos t-3e^{-3t}\sin t$$

***Example* 16** Find the inverse transform of $\frac{s}{(s-3)^5}$.

*Solution*

$$\frac{s}{(s-3)^5}=\frac{s-3+3}{(s-3)^5}=\frac{1}{(s-3)^4}+\frac{3}{(s-3)^5}$$

$$L^{-1}\left[\frac{s}{(s-3)^5}\right]=L^{-1}\left[\frac{1}{(s-3)^4}\right]+L^{-1}\left[\frac{3}{(s-3)^5}\right]$$

$$=e^{3t}\frac{t^3}{3!}+3e^{3t}\frac{t^4}{4!}=\frac{t^3e^{3t}}{3!}\left[1+\frac{3t}{4}\right]$$

***Example* 17** Find the inverse transform of $\frac{1+2s}{(s+2)^2(s-1)^2}$

*Solution* By the method of partial fraction, we obtain

$$\frac{1+2s}{(s+2)^2(s-1)^2}=\frac{1}{3}\frac{1}{(s-1)^2}-\frac{1}{3}\frac{1}{(s+2)^2}.$$

$$\begin{aligned}L^{-1}\left[\frac{1+2s}{(s+2)^2(s-1)^2}\right] &= \frac{1}{3}L^{-1}\frac{1}{(s-1)^2}-\frac{1}{3}L^{-1}\frac{1}{(s+2)^2}\\ &= \frac{1}{3}e^tt-\frac{1}{3}e^{-2t}t=\frac{t}{3}\left[e^t-e^{-2t}\right].\end{aligned}$$

***Example* 18** Find $\mathrm{L}^{-1}\left[\frac{e^{-3s}}{(s-1)^4}\right]$

Solution

By Corollary 2 to Theorem 4,

$$\mathrm{L}^{-1}\left[\frac{1}{(s-1)^4}\right] = e^t\mathrm{L}^{-1}\left[\frac{1}{s^4}\right] = e^t\frac{t^3}{3!} = \frac{t^3e^t}{6}.$$

Letting $F(s) = \frac{1}{(s-1)^4}$, we have $f(t) = \frac{t^3e^t}{6}$, and hence Corollary 1 to Theorem 1(*Shifting on the t-axis*) , gives

$$\begin{aligned}\mathrm{L}^{-1}\left[\frac{e^{-3s}}{(s-1)^4}\right] &= u_3(t)f(t-3)\\ &= \frac{1}{6}u_3(t)\,(t-3)^3e^{t-3}\\ &= \begin{cases}\frac{1}{6}(t-3)^3e^{t-3}, & t>3\\ 0, \ t\leq 3\end{cases}\end{aligned}$$

***Example*** **19** Solve the initial value problem

$$y'' + 2y' + 2y = r(t),$$

$$r(t) = \begin{cases} 10\sin 2t & \text{if } 0 < t < \pi, \\ 0 & \text{if } t > \pi\end{cases}$$

$$y(0) = 1, \quad y'(0) = -5.$$

*Solution*

The given differential equation is

$$y'' + 2y' + 2y = [u(t-0) - u(t-\pi)]10\sin 2t.$$

Taking Laplace transforms on both sides, we obtain

$$L[y'' + 2y' + 2y] = L\left\{[u(t) - u(t-\pi)]10\sin 2t\right\}.$$

That is,

$$(s^2Y - s + 5) + 2(sY - 1) + 2Y = L\left\{u(t)10\sin 2t\right\} + L\left\{u(t-\pi)10\sin 2t\right\} \tag{19.10}$$

Using the result (Second Shifting theorem; Shifting on the $t$-axis)

$$\mathrm{L}\ [f(t-a)u(t-a)] = e^{-as}\mathrm{L}[f(t)] = e^{-as}F(s), \quad \text{for } s > a,$$

we have

$$L\left\{u(t)\sin 2t\right\} = L\left\{u(t-0)\sin 2(t-0)\right\}$$

$$= e^{-0s}\mathrm{L}[\sin 2t]$$

$$= \frac{2}{s^2+4}.$$

Also,

$$\begin{aligned} L\left\{u(t-\pi)\sin 2t\right\} &= L\left\{u(t-\pi)\sin 2(t-\pi)\right\}, \\ &\quad \text{since } \sin 2(t-\pi) = \sin(2t - 2\pi) = \sin 2t \\ &= e^{-\pi s}\mathrm{L}[\sin 2t] \\ &= e^{-\pi s}\frac{2}{s^2+4}. \end{aligned}$$

Hence (19.10) becomes

$$(s^2Y - s + 5) + 2(sY - 1) + 2Y = 10 \cdot \frac{2}{s^2+4}(1 - e^{-\pi s}).$$

i.e.,

$$(s^2 + 2s + 2)Y - s + 3 = \frac{20}{s^2+4}(1 - e^{-\pi s})$$

i.e.,

$$(s^2 + 2s + 2)Y = \frac{20}{s^2+4}(1 - e^{-\pi s}) + s - 3.$$

Hence,

$$Y = \frac{20}{(s^2+4)(s^2+2s+2)} - \frac{20e^{-\pi s}}{(s^2+4)(s^2+2s+2)} + \frac{s-3}{s^2+2s+2} \tag{19.11}$$

In the ***first fraction*** in (19.11) $s^2+4$ contributes unrepeated[1] complex roots $2i$ and $-2i$; also $s^2+2s+2$ contributes unrepeated complex roots; hence the first fraction in (19.11) has a partial fraction representation

$$\frac{20}{(s^2+4)(s^2+2s+2)} = \frac{As+B}{s^2+4} + \frac{Ms+N}{s^2+2s+2}.$$

Multiplication by the common denominator gives

$$20 = (As+B)(s^2+2s+2) + (Ms+N)(s^2+4).$$

---

[1]Ref. the section **Unrepeated Complex Factors** $(s-\mathrm{a})(s-\bar{\mathrm{a}})$ in the case of partial fractions in the chapter “Laplace and Inverse Laplace Transforms”.

We determine $A$, $B$, $M$, $N$. Equating the coefficients of each power of $s$ on both sides gives the four equations

(a) $[s^3]: \quad 0 = A + M$

(b) $[s^2]: \quad 0 = 2A + B + N$

(c) $[s]: \quad 0 = 2A + 2B + 4M$

(d) $[s^0]: \quad 20 = 2B + 4N$

We can solve this, for instance, obtaining $M = -A$ from (a), then $A = B$ from (c), then $N = -3A$ from (b), and finally $A = -2$ from (d). Hence $A = -2$, $B = -2$, $M = 2$, $N = 6$, and the first fraction in (19.11) has the representation

$$\frac{-2s-2}{s^2+4} + \frac{2s+6}{(s+1)^2+1}$$

or

$$\frac{-2s-2}{s^2+4} + \frac{2(s+1)+4}{(s+1)^2+1}$$

or

$$-2\frac{s}{s^2+2^2} - \frac{2}{s^2+2^2} + 2 \cdot \frac{s+1}{(s+1)^2+1} + 4 \cdot \frac{1}{(s+1)^2+1}.$$

Its inverse transform (in the last two summands using First Shifting Theorem) is

$$-2\cos 2t - \sin 2t + e^{-t}(2\cos t + 4\sin t). \tag{19.12}$$

In the ***second fraction*** in (19.11) taken with the ***minus sign*** we have the factor $e^{-\pi s}$, so that from (19.12) and the second shifting theorem (shifting on the $t$-axis) [i.e., using

$$\mathrm{L}^{-1}\left[e^{-as}F(s)\right] = f\,(t\ -\ a)u(t-a)],$$

we get the inverse transform [*Attention!* The minus sign is included in the following:]

$$-\{-2\cos 2(t-\pi) - \sin 2(t-\pi) + \\ e^{-(t-\pi)}\left[2\cos(t-\pi) + 4\sin(t-\pi)\right]u(t-\pi)\}$$

$$\stackrel{*}{=} -\left\{-2\cos 2t - \sin 2t + e^{-t}e^{\pi}[-2\cos t - 4\sin t]\right\}u(t-\pi)$$

$$= \left\{2\cos 2t + \sin 2t + e^{-t}e^{\pi}[2\cos t + 4\sin t]\right\}u(t-\pi). \qquad (19.13)$$

2

[2]In the equality $\stackrel{*}{=}$ above, we have used the following trigonometric identities:

$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$
$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$

which in particular gives

$$\cos 2(t-\pi) = \cos(2t-2\pi) = \cos 2t;$$

$$\sin 2(t-\pi) = \sin(2t-2\pi) = \sin 2t;$$

$$\cos(t-\pi) = -\cos t; \sin(t-\pi) = -\sin t;$$

Thus, **for $t > \pi$,** since $u(t-\pi) = 1$, (19.13) gives

$$2\cos 2t + \sin 2t + e^{-t}e^{\pi}[2\cos t + 4\sin t] \qquad (19.14)$$

Now, in the ***last fraction*** in (19.11) applying Corollary 2 to First Shifting Theorem (Shifting on the $s$-axis) in page 331, i.e., using

$$\mathrm{L}^{-1}\left[F(s-a)\right] = e^{at}\mathrm{L}^{-1}\left[F(s)\right],$$

we obtain

$$\begin{aligned} L^{-1}\left[\frac{s-3}{s^2+2s+2}\right] &= L^{-1}\left\{\frac{s+1-4}{(s+1)^2+1}\right\} \\ &= L^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} - 4L^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} \\ &= e^{-t}(\cos t - 4\sin t) \qquad (19.15) \end{aligned}$$

We conclude that:

1. The solution of the given initial value problem **for** $0 < t < \pi$, is the sum of (19.12) and (19.15); and is

$$y(t) = 3e^{-t}\cos t - 2\cos 2t - \sin 2t \text{ for } 0 < t < \pi$$

1. The solution of the given initial value problem **for**$t > \pi$, is the sum of (19.12), (19.14) and (19.15); and is $y(t) = e^{-t}[(3 + 2e^{\pi})\cos t + 4e^{\pi}\sin t]$ for$t > \pi$.

**Exercises** In Exercises 1-18, find the inverse Laplace transform of the given function.

1. $\frac{n\pi}{(s+2)^2+n^2\pi^2}$
2. $\frac{a_1}{(s-3)^2} + \frac{2a_2}{(s-3)^3}$
3. $\frac{s}{(s+3)^2+1}$
4. $\frac{as+b}{(s+c)^2+\omega^2}$
5. $\frac{1}{(s+1)(s^2+2s+2)}$
6. $\frac{3s^2+16s+26}{s(s^2+4s+13)}$
7. $\frac{2(s-a)}{(s-a)^2+b^2} + \frac{8-6s}{16s^2+9}$
8. $\frac{1}{s(s+2)^3}$
9. $\frac{2s^2-4}{(s+1)(s-2)(s-3)}$
10. $\frac{3s-8}{s^2-4s+20}$
11. $\frac{6s-4}{s^2-4s+20}$
12. $\frac{1}{(s-5)^{10}}$
13. $\frac{s-3}{(s-3)^2+5^2}$
14. $\frac{s+2}{(s+2)^2+3^2}$
15. $\frac{s}{(s-3)^8}$
16. $\frac{s}{(s+8)^2+7^2}$
17. $\frac{1}{(s+1)^2}$
18. $\frac{2}{(s-a)^3}$

***Answers***

1. $e^{-2t}\sin\pi t$
2. $e^{3t}\left(a_1 t + a_2 t^2\right)$
3. $e^{-3t}(\cos t - 3\sin t)$
4. $e^{-ct}\left[a\cos\omega t + \frac{b-ac}{\omega}\sin\omega t\right]$
5. $e^{-t}[1-\cos t]$
6. $2+e^{-2t}\cos 3t+2e^{-2t}\sin 3t$
7. $2e^{at}\cos bt + \frac{1}{2}\sin\frac{3t}{4} - \frac{3}{8}\cos\frac{3t}{4}$
8. $\frac{1}{8}-\frac{1}{8}e^{-2t}-\frac{1}{4}te^{-2t}-\frac{1}{4}t^2e^{-2t}$
9. $-\frac{1}{6}e^{-t} - \frac{4}{3}e^{2t} + \frac{7}{2}e^{3t}$
10. $3e^{2t}\cos 4t - \frac{1}{2}e^{2t}\sin 4t$
11. $6e^{2t}\cos 4t + 2e^{2t}\sin 4t$
12. $e^{5t}\frac{t^9}{9!}$
13. $e^{3t}\cos 5t$
14. $e^{-2t}\cos 3t$

15. $e^{3t}\left[\frac{t^6}{6!}+3\frac{t^7}{7!}\right]$

16. $e^{-8t}(\cos 7t-\frac{8}{7}\sin 7t)$

17. $e^{-t}t$

18. $t^2e^{at}$

**Table:** *Some elementary functions F(s) and their inverse Laplace transforms*

| | $F(s)$ | $f(t) = L^{-1}[F(s)]$ | | $F(s)$ | $f(t) = L^{-1}[F(s)]$ |
|---|---|---|---|---|---|
| 1 | 0 | 0 | 8 | $\frac{1}{s-i\,a}$ | $e^{iat}$ |
| 2 | $\frac{1}{s}$ | *1* | 9 | $\frac{s}{s^2+a^2}$ | $\cos at$ |
| 3 | $\frac{1}{s^2}$ | $t$ | 10 | $\frac{1}{s^2+a^2}$ | $\frac{\sin at}{a}$ |
| 4 | $\frac{1}{s^3}$ | $\frac{t^2}{2!}$ | 11 | $\frac{s}{s^2-a^2}$ | $\cosh at$ |
| 5 | $\frac{1}{s^{n+1}}$ | $\frac{t^n}{n!}$ $n=1,2,\ldots$ | 12 | $\frac{1}{s^2-a^2}$ | $\frac{1}{a}\sinh at$ |
| 6 | $\frac{1}{s^{a+1}}$ | $\frac{t^a}{\Gamma(a+1)}$ *a is positive* | 13 | $\frac{s-a}{(s-a)^2+\omega^2}$ | $e^{at}\cos\omega t$ |
| 7 | $\frac{1}{s-a}$ | $e^{at}$ | | $\frac{\omega}{(s-a)^2+\omega^2}$ | $e^{at}\sin\omega t$ |

## 19.2 Impulse Functions. Dirac's Delta Function

Phenomena of an impulsive nature, such as the action of very large forces over very short intervals of time, are of great practical interest, since they arise in various applications. This situation occurs, for instance, when a cricket ball is hit, a system is given a blow by a hammer, and so on. Such problems often lead to differential equation of the form

$$ay'' + by' + cy = g(t),$$

where $g(t)$ is large during a short interval $t_0 - \tau < t < t_0 + \tau$ and is otherwise zero.

We now show how to solve problems involving short impulses by Laplace transforms.

In mechanics, the **impulse** $I(\tau)$ of a force $g(t)$ over the time interval $t_0 - \tau < t < t_0 + \tau$ is defined to be the integral of $g(t)$ from $t_0 - \tau$ to $t_0 + \tau$. i.e.,

$$I(\tau) = \int_{t_0-\tau}^{t_0+\tau} g(t)\, dt,$$

or, since $g(t) = 0$ outside of the interval $(t_0 - \tau,\ t_0 + \tau)$,

$$I(\tau) = \int_{-\infty}^{\infty} g(t)\, dt.$$

Impulse is a measure of the strength of the forcing function.

The analog for an electric circuit is the integral of the electro motive force applied to the circuit, integrated from$t_0 - \tau$ to $t_0 + \tau$. Of particular practical interest is the case of a very short $\tau$ (and its limit $\tau \to 0$), that is, the impulse of a force acting only for an instant. To handle the case, we consider the function (with $t_0 = 0$)

$$g(t) = d_\tau(t) = \begin{cases} \frac{1}{2\tau} & \text{if } -\tau < t < \tau \\ \\ 0 & \text{otherwise} \end{cases} \tag{19.16}$$

where $\tau$ is small positive constant.

Its impulse $I(\tau)$ is 1, since

$$I(\tau) = \int_{-\infty}^{\infty} g(t)\,dt = \int_{-\tau}^{\tau} \frac{1}{2\tau} dt = \frac{1}{2\tau}[t]_{-\tau}^{\tau} = 1.$$

**Attention!** In the above, $\tau > 0$ plays an important role, because only then $[t]_{-\tau}^{\tau} = 2\tau$.

We note that

$$\lim_{\tau\to 0} d_\tau(1) = 0, \qquad \tau \neq 0. \tag{19.17}$$

Also, since

$I(\tau) = 1$ for each $\tau \neq 0$

it follows that

$$\lim_{\tau\to 0} I(\tau) = 1. \tag{19.18}$$

Equations (19.17) and (19.18) can be used to define *unit impulse function*$\delta$, which imparts an impulse of magnitude one at $t = 0$ but is 0 for all values of $t$ other than 0. i.e., the **unit impulse function** $\delta$ is defined to have the properties

$$\delta(t) = 0, \quad t \neq 0 \tag{19.19}$$

and

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1. \tag{19.20}$$

**Attention!** There is no ordinary function of the kind studied in calculus that satisfies both (19.19) and (19.20), because an ordinary function that is every where zero except at a single point must have the integral 0, not 1. Though $\delta(t)$ is not a function in the ordinary sense, it is a so-called "generalized function".

Nevertheless, in impulse problems it is convenient to operate on $\delta(t)$ as a 'function'. The "function" $\delta$ defined by (19.18) and (19.19) is usually called the **Dirac delta function.** $\delta(t)$ corresponds to impulse at $t = 0$. A unit impulse at an arbitrary point $t = t_0$ is given by $\delta(t - t_0)$. From equations (19.19) and (19.20) it follows that

$$\delta(t - t_0) = 0, \qquad t \neq t_0$$

and

$$\int_{-\infty}^{\infty} \delta(t - t_0) dt = 1.$$

The delta function does not satisfy the conditions of Existence Theorem for Laplace Transforms, but its Laplace transform can be formally defined. Since $\delta(t)$ is defined as the limit of $d_\tau(t)$ as $\tau \to 0$,it is natural to define the Laplace transform of $\delta$ as a similar limit of the transform of $d_\tau$. In particular, we will assume that $t_0 > 0$ and define $L\{\delta(t - t_0)\}$ by the equation

$$L\{\delta(t - t_0)\} = \lim_{\tau \to 0} L\{d_\tau(t - t_0)\}. \tag{19.21}$$

To evaluate the limit in Eq. (19.21), we first observe that if $\tau < t_0$, which must eventually be the case as $\tau \to 0$, then $t_0 - \tau > 0$. Since $d_\tau(t - t_0)$is nonzero only in the interval from $t_0 - \tau$ to $t_0 + \tau$, we have

$$L\{d_\tau(t - t_0)\} = \int_0^\infty e^{-st} d_\tau(t - t_0)\ dt$$

$$= \int_{t_0-\tau}^{t_0+\tau} e^{-st} d_\tau(t - t_0) dt.$$

Substituting for $d_\tau(t - t_0)$from Eq. (19.16), we obtain

$$L\{d_\tau(t - t_0)\} = \frac{1}{2\tau} \int_{t_0-\tau}^{t_0+\tau} e^{-st} dt = -\frac{1}{2s\tau} \left[e^{-st}\right]_{t=t_0-\tau}^{t=t_0+\tau}$$

$$= \frac{1}{2s} e^{-st_0} (e^{s\tau} - e^{-s\tau})$$

or

$$L\{d_\tau(t - t_0)\} = \frac{\sinh\ s\tau}{s\tau} e^{-st_0}. \tag{19.22}$$

The quotient $(\sinh\ s\tau)/s\tau$ is indeterminate as $\tau \to 0$, but its limit can be evaluated by L'Hopital's rule.

$\lim\limits_{\tau \to 0} \frac{\sinh\ s\tau}{s\tau} = \lim\limits_{\tau \to 0} \frac{s \cosh\ s\tau}{s}$,by applying L'Hopital's rule.

$$= \lim_{\tau \to 0} \cosh\ s\tau = 1.$$

Then from Eq. (19.21) and (19.22) it follows that

$$L\{\delta(t - t_0)\} = e^{-st_0}. \tag{19.23}$$

Equation (19.23) defines $L\{\delta(t - t_0)\}$for any $t_0 > 0$. We extend this result, to allow $t_0$ to be zero, by letting $t_0 \to 0$ on the right side of Eq. (19.23); thus

$$L\{\delta(t)\} = \lim_{t_0 \to 0} e^{-st_0} = 1 \tag{19.24}$$

In a similar way it is possible to define the integral of the product of the delta function and any continuous function *f*. We have

$$\int_{-\infty}^{\infty} \delta(t-t_0)f(t)\,dt = \lim_{\tau \to 0} \int_{-\infty}^{\infty} d_\tau(t-t_0)f(t)dt \tag{19.25}$$

Now

$\int_{-\infty}^{\infty} d_\tau(t-t_0)f(t)dt = \frac{1}{2\tau}\int_{t_0-\tau}^{t_0+\tau} f(t)dt$, using the definition (19.16) of $d_\tau(t)$

$= \frac{1}{2\tau} \cdot 2\tau \cdot f(t^*)$, using the mean value theorem for integrals.

$$= f(t^*),$$

where $t_0 - \tau < t^* < t_0 + \tau$. Hence $t^* \to t_0$ as $\tau \to 0$, and it follows from Eq.(19.25) that

$$\int_{-\infty}^{\infty} \delta(t-t_0)f(t)\,dt = f(t_0). \tag{19.26}$$

***Example* 20** Solve

$$y'' + 3y' + 2y = \delta(t-1), y(0) = 0, \quad y'(0) = 0$$

*Solution*

Here $t_0 = 1$. Hence Eq.(19.23) gives $L[\delta(t-1)] = e^{-s\times 1} = e^{-s}$.

Thus

$$L[y'' + 3y' + 2y] = L[\delta(t-1)]$$

gives

$$s^2Y + 3sY + 2Y = e^{-s}$$

or

$$Y = \frac{1}{s^2+3s+2}e^{-s} = \frac{e^{-s}}{(s+1)(s+2)} = \left(\frac{1}{s+1} - \frac{1}{s+2}\right)e^{-s}.$$

Taking $F(s)= \frac{1}{s+1} - \frac{1}{s+2}$, we obtain

$$f(t) = L^{-1}[F(s)] = L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{1}{s+2}\right] = e^{-t} - e^{-2t}.$$

Hence, by (Shifting on the $t$-axis) Theorem, we have

$$y(t) = L^{-1}\left\{e^{-s}F(s)\right\} = u_1(t)f(t-1)$$

$$= \begin{cases} 0 & (0 \le t \le 1) \\ e^{-(t-1)} - e^{-2(t-1)} & (t > 1) \end{cases}$$

***Example* 21** Solve

$$y'' + 2y' + 2y = \delta(t-\pi), \qquad y(0) = 1, y'(0) = 0.$$

*Solution*

Taking$t_0 = \pi$, Eq.(19.23) gives

$$L[\delta(t-\pi)] = e^{-\pi s}.$$

Now taking Laplace transform on both sides, the differential equation yields

$$(s^2+2s+2)\ Y(s) - s - 2 = e^{-\pi s}.$$

Hence

$$Y(s) = \frac{s+2}{s^2+2s+2} + \frac{e^{-\pi s}}{s^2+2s+2}.$$

Now

$$\frac{s+2}{s^2+2s+2} = \frac{s+2}{(s+1)^2+1} = \frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}$$

Since $L^{-1}\left[\frac{s}{s^2+1}\right] = \cos t$ and $L^{-1}\left[\frac{1}{s^2+1}\right] = \sin t$, we have

$$L^{-1}\left[\frac{s+1}{(s+1)^2+1}\right] = L^{-1}\left[\frac{s-(-1)}{[s-(-1)]^2+1}\right] = e^{-t}\cos t$$

and $L^{-1}\left[\frac{1}{(s+1)^2+1}\right] = L^{-1}\left[\frac{1}{[s-(-1)]^2+1}\right] = e^{-t}\sin t.$

Hence

$$L^{-1}\left[\frac{s+2}{s^2+2s+2}\right] = e^{-t}\cos t + e^{-t}\sin t.$$

Taking

$$F(s) = \frac{1}{s^2+2s+2} = \frac{1}{(s+1)^2+1}$$

and noting the already obtained fact that

$$f(t) = L^{-1}[F(s)] = e^{-t}\sin t,$$

we have

$$\begin{aligned} L^{-1}[e^{-\pi s}\cdot F(s)] &= u_\pi(t)\, f(t-\pi) \\ &= u_\pi(t)\, e^{-(t-\pi)}\sin(t-\pi) \end{aligned}$$

Hence

$$y(t) = L^{-1}[Y(s)] = e^{-t}\cos t + e^{-t}\sin t + u_\pi(t)\, e^{-(t-\pi)}\sin(t-\pi)$$

Chapter 20

# Differentiation and Integration of Transforms

In this chapter we consider differentiation and integration of transforms and finding out the corresponding operations for original functions.

## 20.1 Differentiation of Transforms

**Theorem 1 (Differentiation of Transforms)**
If $\mathrm{L}[f(t)] = F(s)$, then

$$\mathrm{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s) = (-1)^n F^{(n)}(s) \ \ (n = 1,\ 2,\ 3,\ \ldots) \tag{20.1}$$

**Proof** By the definition of Laplace transform we have

$$F(s) = \mathrm{L}[f(t)] = \int_0^\infty e^{-st} f(t)dt.$$

Hence

$$\frac{d}{ds}F(s) = \frac{d}{ds}\left[\int_0^\infty e^{-st} f(t)dt\right].$$

On the RHS we have to perform *first* integration and *then* a differentiation. Under certain conditions, it is permissible to interchange these operators.

Let us suppose that these conditions are satisfied in this case. Moreover, when we take the differentiation under the integral sign, since there are two variables $s$ and $t$, we shall use the symbol $\frac{\partial}{\partial s}$. Thus

$$\begin{aligned}
\frac{d}{ds}F(s) &= \int_0^\infty \frac{\partial}{\partial s}\left\{e^{-st} f(t)dt\right\} \\
&= \int_0^\infty e^{-st}(-t)f(t)dt = -\int_0^\infty e^{-st}[t\ f(t)]dt \\
&= -\mathrm{L}\left[tf(t)\right], \text{by the definition of Laplace transform.}
\end{aligned}$$

Thus,

$$\mathrm{L}[t\, f(t)] \;=\; -\frac{d}{ds}\left[F(s)\right] = -F'(s) \qquad (20.2)$$

By a repeated application, it can be proved that

$$\mathrm{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}F(s) = (-1)^n F^{(n)}(s), \;\; n \;= 1, 2, ...$$

***Example* 1** Find $\mathrm{L}[t\ e^{at}]$

*Solution*

Take $f(t) = e^{at}$, then

$F(s) = \mathrm{L}\left[f\ (t)\right] = \mathrm{L}\left[e^{at}\right] = \frac{1}{s-a};\ s > a$

Hence using (20.2) above we have

$$\mathrm{L}[te^{at}] = -\frac{d}{ds}F(s) = -\frac{d}{ds}\left(\frac{1}{s-a}\right) = \frac{1}{(s-a)^2},\ \ s > a;$$

***Example* 2** Show that $\mathrm{L}[t^n e^{at}] = \frac{n!}{(s-a)^{n+1}}$

*Solution*

We know that

$$\mathrm{L}[e^{at}] = \frac{1}{s-a},\ \ s > a.$$

Hence, using theorem, we have

$$\begin{aligned}
\mathrm{L}[t^n e^{at}] &= (-1)^n \frac{d^n}{ds^n}\left(\frac{1}{s-a}\right) \\
&= (-1)^n(-1)\frac{d^{n-1}}{ds^{n-1}}\left(\frac{1}{(s-a)^2}\right) \\
&\vdots \\
&= (-1)^n(-1)^n \frac{n\,!}{(s-a)^{n+1}} \\
&= \frac{n!}{(s-a)^{n+1}}.
\end{aligned}$$

In particular,

$$\mathrm{L}[te^{at}] = \frac{1}{(s-a)^2},\ \ s > a;$$

$$\mathrm{L}[t^2 e^{at}] = \frac{2}{(s-a)^3}, \quad s > a.$$

***Example* 3** Show that $\mathrm{L}[t^2 e^{at}] = \frac{2!}{(s-a)^3}$.

*Solution*

This is a particular case of the above example. If we prove independently, then

$$\begin{aligned}
\mathrm{L}[t^2 e^{at}] &= (-1)^2 \frac{d^2}{ds^2}\left(F(s)\right), \text{ where } F(s) = \mathrm{L}[e^{at}] \\
&= (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{s-a}\right) \\
&= \frac{2!}{(s-a)^3}.
\end{aligned}$$

***Example* 4** Find $\mathrm{L}[t \sin t]$

*Solution*

$\mathrm{L}[\sin t] = \frac{1}{\mathrm{s}^2+1}$ implies that

$$\mathrm{L}[t \sin t] = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2}.$$

***Example* 5** Prove that $\mathrm{L}[t \sin at] = \frac{2as}{(s^2+a^2)^2}$.

*Solution*

Proceed as in the previous example.

***Example* 6** Find $\mathrm{L}[t^2 \cos t]$

*Solution*

We know that $\mathrm{L}[\cos t] = \frac{s}{s^2+1}$. Hence, by the theorem,

$$\begin{aligned}
\mathrm{L}[t^2\cos t] &= (-1)^2 \frac{d^2}{ds^2}\left(\frac{s}{s^2+1}\right), \\
&= \frac{d}{ds}\left(\frac{(s^2+1) - 2s \times s}{(s^2+1)^2}\right), \quad \text{using} \\
&\quad \text{quotient rule of differentiation} \\
&= \frac{d}{ds}\left(\frac{1-s^2}{(s^2+1)^2}\right) \\
&= \frac{(s^2+1)^2(-2s) - (1-s^2)(s^2+1)4s}{(s^2+1)^4}, \quad \text{again using} \\
&\quad \text{quotient rule of differentiation} \\
&= \frac{2s^3 - 6s}{(s^2+1)^3}, \quad \text{on simplification}
\end{aligned}$$

***Example* 7** Find $\mathrm{L}[t^2 \cosh \pi t]$.

*Solution*

$$\begin{aligned}
\mathrm{L}[t^2\cosh \pi t] &= (-1)^2 \frac{d^2}{ds^2}\left(\frac{s}{s^2-\pi^2}\right), \\
&= \frac{2s^3 + 6\pi^2 s}{(s^2-\pi^2)^3},
\end{aligned}$$

on simplification.

***Example* 8** Find $\mathrm{L}[te^{-t}\sin t]$

*Solution*

We have

$$\mathrm{L}[\sin t] = \frac{1}{s^2+1}$$

and hence by shifting on the $s$-axis theorem,

$$\mathrm{L}[e^{-t}\sin t] = \frac{1}{(s+1)^2+1}.$$

Now using Theorem 1 this implies

$$\begin{aligned}\mathrm{L}[te^{-t}\sin t] &= -\frac{d}{ds}\left(\frac{1}{(s+1)^2+1}\right)\\ &= \frac{2(s+1)}{\left((s+1)^2+1\right)^2}, \text{ using quotient rule}\\ &= \frac{2(s+1)}{(s^2+2s+2)^2}.\end{aligned}$$

***Example* 9** Find $\mathrm{L}[te^{2t}\cos 5t]$.

*Solution*

We have $\mathrm{L}[\cos 5t] = \frac{s}{s^2+25}$ and hence

$$\mathrm{L}[e^{2t}\cos 5t] = \frac{s-2}{(s-2)^2+25}.$$

Now this implies by Theorem 1 that

$$\begin{aligned}\mathrm{L}[te^{2t}\cos 5t] &= -\frac{d}{ds}\left(\frac{s-2}{(s-2)^2+25}\right)\\ &= \frac{(s-2)^2-25}{\left[(s-2)^2+25\right]^2},\end{aligned}$$

on simplification.

## 20.2 Exercises

In Exercises 1-7, find the Laplace transform.

1. $t\ \cos 2t$
2. $te^{2t}$
3. $t\ \cosh t$
4. $t^2 e^t$
5. $t\ \sinh 2t$
6. $t^2 \sinh 2t$
7. $t^2\ \cos \omega t$

8. Using Differentiation of Transforms Theorem, derive the following formulae:

   $(i)$ $\mathrm{L}^{-1}\left[\frac{s}{(s^2+\beta^2)^2}\right] = \frac{1}{2\beta}(t \sin \beta t)$

   $(ii)$ $\mathrm{L}^{-1}\left[\frac{1}{(s^2+\beta^2)^2}\right] = \frac{1}{2\beta^3}(\sin \beta t - \beta t \cos \beta t)$

   $(iii)$ $\mathrm{L}^{-1}\left[\frac{s^2}{(s^2+\beta^2)^2}\right] = \frac{1}{2\beta}(\sin \beta t + \beta t \cos \beta t)$

### 20.2.1 Answers

1. $\frac{s^2-4}{(s^2+4)^2}$
2. $\frac{1}{(s-2)^2}$
3. $\frac{s^2+1}{(s^2-1)^2}$
4. $\frac{2}{(s-1)^3}$
5. $\frac{4s}{(s^2-4)^2}$
6. $\frac{4\left(4+3s^2\right)}{(s^2-4)^3}$
7. $\frac{2s(s^2-3\omega^2)}{(s^2+\omega^2)^3}$

## 20.3 Integration of Transforms

**Theorem 2 (Integration of Transform)** If $f(t)$ satisfies the conditions of the existence of Laplace transform theorem and the limit of $\frac{f(t)}{t}$ as $t$ approaches to 0 from the right exists, then

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(u)du \;\; (\, s > \gamma \,) \tag{20.3}$$

**Proof** From the definition of Laplace transform it follows that

$$\int_s^\infty F(u)du = \int_s^\infty \left[\int_0^\infty e^{-ut} f(t)dt\right] du$$

Under the assumptions of the theorem it is possible to interchange the order of integration, so that

$$\int_s^\infty F(u)du = \int_0^\infty \left[\int_s^\infty e^{-ut} f(t)du\right] dt = \int_0^\infty f(t) \left[\int_s^\infty e^{-ut} du\right] dt$$

The integral over $u$ on the right equals$\frac{e^{-st}}{t}$, when $s > \gamma$ and therefore,

$$\int_s^\infty F(u)du = \int_0^\infty e^{-st} \frac{f(t)}{t} dt = L\left[\frac{f(t)}{t}\right], \quad s > \gamma$$

***Example* 10** Find $L\left[\frac{1-e^t}{t}\right]$

*Solution*

Here take $f(t) = 1 - e^t$, then

$$F(s) = L[f(t)] = L[1] - L[e^t] = \frac{1}{s} - \frac{1}{s-1}$$

$$\begin{aligned}
L\left[\frac{1-e^t}{t}\right] &= L\left[\frac{f(t)}{t}\right] \\
&= \int_s^\infty F(u)du, \text{ using Theorem 2} \\
&= \int_s^\infty \left(\frac{1}{u} - \frac{1}{u-1}\right) du \\
&= [\log u - \log(u-1)]_s^\infty \\
&= \left[\log \frac{u}{u-1}\right]_s^\infty = \log 1 - \log \frac{s}{s-1} \\
&= -\log \frac{s}{s-1} = \log \frac{s-1}{s}.
\end{aligned}$$

***Example* 11** Find $L\left[\frac{\sin at}{t}\right]$

*Solution*

Since $L[\sin at] = \frac{a}{s^2+a^2}$,

$$L\left[\frac{\sin at}{t}\right] = \int_s^\infty \frac{a}{u^2+a^2} du = \left[\tan^{-1}\frac{u}{a}\right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}\frac{s}{a} = \cot^{-1}\frac{s}{a}$$

**Inverse Transform: Corollary to (3)**

Using (20.3) above, we have

$$L^{-1}\left[\int_s^\infty F(u)du\right] = \frac{f(t)}{t}. \tag{20.4}$$

The following result is needed in the development of the coming example.

**RESULT Show that** if $\int_s^\infty F(u)du = g(s)$, then $F(u) = -\frac{d}{du}g(u)$.

*Proof*

$$\begin{aligned}
-\frac{d}{du}g(u) &= -\lim_{h\to 0}\frac{g(u+h)-g(u)}{h} \\
&= -\lim_{h\to 0}\frac{\int_{u+h}^{\infty}F(r)dr-\int_{u}^{\infty}F(r)dr}{h} \\
&= \lim_{h\to 0}\frac{\int_{u}^{\infty}F(r)dr-\int_{u+h}^{\infty}F(r)dr}{h} \\
&= \lim_{h\to 0}\frac{\int_{u}^{u+h}F(r)dr}{h} \\
&= F(u).
\end{aligned}$$

***Example* 12** Find the inverse transform of the function $\log_e(1+\frac{\omega^2}{s^2})$

*Solution*

Let $\int_s^\infty F(u)du = \log_e\left(1+\frac{\omega^2}{s^2}\right)$

Then using the above Result,

$$F(u) = -\frac{d}{du}\log_e\left(1+\frac{\omega^2}{u^2}\right) = -\frac{d}{du}\log_e\left(\frac{u^2+\omega^2}{u^2}\right)$$

$$= \frac{d}{du}\left(2\log u - \log(u^2+\omega^2)\right) = \frac{2}{u} - \frac{2u}{u^2+\omega^2}$$

$$\therefore f(t) = \mathrm{L}^{-1}\left[F(s)\right] = \mathrm{L}^{-1}\left[\frac{2}{s}-\frac{2s}{s^2+\omega^2}\right] = 2-2\cos\omega t.$$

This function satisfies the conditions under which (20.3) holds. Hence using (20.4), we obtain

$$\mathrm{L}^{-1}\left[\log\left(1+\frac{\omega^2}{s^2}\right)\right] = \mathrm{L}^{-1}\left[\int_s^\infty F(u)du\right] = \frac{f(t)}{t} = \frac{2}{t}(1-\cos\omega t).$$

**Exercises**

In Exercises 1-6, find the inverse Laplace transform of the given function.

1. $\log_e \frac{s+a}{s+b}$
2. $\log_e \frac{s}{s-1}$
3. $\log_e \frac{s^2+1}{(s-1)^2}$
4. $\cot^{-1} \frac{s}{\omega}$
5. $\cot^{-1}(s+1)$
6. $\tan^{-1} \frac{1}{s}$

***Answers***

1. $\frac{e^{-bt}-e^{-at}}{t}$
2. $\frac{e^{t}-1}{t}$
3. $\frac{2\left(e^{t}-\cos t\right)}{t}$
4. $\frac{\sin \omega t}{t}$
5. $\frac{e^{-t}\sin t}{t}$
6. $\frac{\sin t}{t}$

# Chapter 21

# Convolution and Integral Equations

## 21.1 Convolution and Integral Equations

An important general property of Laplace transformation has to do with product of transforms. It often happens that we are given two transforms $F(s)$ and $G(s)$ whose inverses $f(t)$ and $g(t)$ we know, and we would like to find the inverse of the product $F(s)G(s)$ from those known inverses $f(t)$and $g(t)$. The inverse is written

$(f * g)(t)$, which is a standard notation and is called the **convolution** of $f$ and $g$. Definition follows:

**Definition** The **convolution** of $f$ and $g$ is a function $h$, usually written $f * g$, defined by

$$h(t) = (f * g)(t) = \int_0^t f(t-u)g(u)du.$$

**Theorem 1 (Properties of Convolution)**
*(i). (Commutativity of convolution)* $f * g = g * f$.
*(ii). (Distributive law)* $f * (g + h) = f * g + f * h$
*(iii). (Associative law)* $(f * g) * h = f * (g * h)$
*(iv).* $f * 0 = 0 * f = 0$.

**Proof** We prove only *(i)* leaving the rest to the exercises.
Put $t - u = v$, then $du = -\ dv$,
$u = 0$ implies $v = t$; $u = t$ implies $v = 0$.
Hence

$$(f * g)(t) = -\int_t^0 f(v)g(t-v)dv = \int_0^t g(t-v)f(v)dv = (g * f)(t).$$

**Theorem 2 (Convolution Theorem)** If $f(t)$ and $g(t)$ are the inverse transforms of *F(s)* and *G(s)* respectively, the inverse transform of the product *F(s)G(s)* is the convolution of $f(t)$and $g(t)$. i.e.,

$$\mathrm{L}^{-1}[F(s)G(s)] = (f * g)(t) \tag{21.1}$$

***Proof of the Convolution Theorem***
By definition

$$\mathrm{L}^{-1}[F(s)] = f(t) \text{ and } \mathrm{L}^{-1}[G(s)] = g(t)$$

and this implies that

$$\mathrm{L}\left[f(t)\right] = F(s) \quad \text{and} \quad \mathrm{L}\left[g(t)\right] = G(s).$$

Then

$$F(s)G(s) = F(s)\mathrm{L}\left[g(t)\right] = F(s)\int_0^\infty e^{-st}g(t)dt = \int_0^\infty e^{-st}F(s)g(t)dt \tag{21.2}$$

But using the definition of $F(s)$,

$$\begin{aligned} e^{-st}F(s) &= e^{-st}\mathrm{L}\left[f(t)\right] = e^{-st}\int_0^\infty e^{-su}f(u)du \\ &= \int_0^\infty e^{-s(u+t)}f(u)du \\ &= \int_t^\infty e^{-sv}f(v-t)dv, \text{ by putting } u+t=v. \end{aligned}$$

Hence (21.2) implies

$$F(s)G(s) = \int_{t=0}^{t=\infty}\int_{v=t}^{v=\infty} e^{-sv}f(v-t)g(t)\,dv\,dt \tag{21.3}$$

Now we assume that the functions $f$ and $g$ are such as to justify a change in the order of integration in (21.3). We note that in (21.3), we first integrate with respect to $v$ from $v = t$ to $v = \infty$ and then integrate with respect to $t$ from $t = 0$ to $t = \infty$. The region of integration $R$ is

$$R: \quad 0 \le t < \infty; \quad t \le v < \infty. \tag{21.4}$$

In order to change the order integration in (21.3), we have to find the alternate form of the region $R$. For this we first draw a rough sketch of the region $R$ using the following facts:

We note that $R$ is the intersection of the following two regions in the $vt$ plane:

(i) the upper half plane in the $vt$- plane (this region corresponds to $0 \leq t < \infty$ .)

(ii) the region to the right of the line $v = t$ (this region corresponds to $t \leq v \leq \infty$ .)

The region $R$ is shown in Fig.21.1 where the shaded portion is extended to infinity in the $vt$-plane.

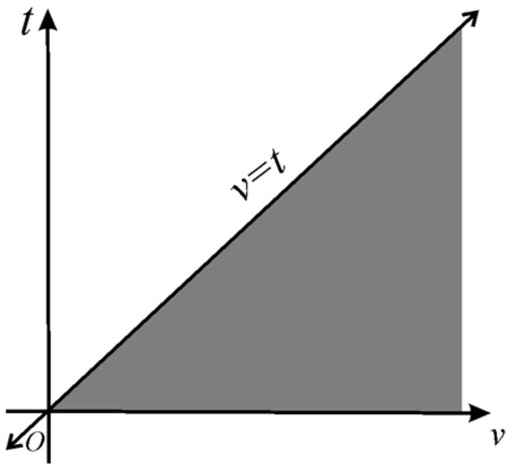


Figure 21.1: The region of integration is the shaded portion that is extended to infinity

Referring to Fig. 21.1, we note that while $v$ varies from 0 to $\infty$, $t$ varies from 0 to $v$. Hence the alternate form of (21.4) is

$$0 \leq v < \infty; \quad 0 \leq t \leq v \tag{21.5}$$

Now using the system of inequalities in (21.5) as the representation of the region of integration, by changing the order of integration, (21.3) yields

$$\begin{aligned} F(s)G(s) &= \int_{v=0}^{v=\infty} \int_{t=0}^{t=v} e^{-sv} f(v-t)g(t)dt\,dv \\ &= \int_{v=0}^{v=\infty} e^{-sv} \left\{ \int_{t=0}^{t=v} f(v-t)g(t)dt \right\} dv \end{aligned} \tag{21.6}$$

Now, by the definition of convolution,

$$\int_{t=0}^{t=v} f(v-t)g(t)dt = (f * g)(v).$$

Hence, (21.6) becomes

$$F(s)G(s) = \int_{v=0}^{v=\infty} e^{-sv}(f * g)(v) = \mathrm{L}[(f * g)(v)].$$

Hence

$$\mathrm{L}^{-1}[F(s)G(s)] = (f * g)(t)$$

This completes the proof.

**Remark to Convolution Theorem**

By the notations as in theorem, (21.1) can be restated as

$$F(s)G(s) = \mathrm{L}\left[(f * g)(t)\right]. \tag{21.7}$$

***Example* 1** Using convolution property, find $\mathrm{L}^{-1}\left[\frac{1}{s(s^2+a^2)}\right]$.

*Solution*

Let $F(s) = \frac{1}{s}$, $G(s) = \frac{1}{s^2+a^2}$, then

$$f(t) = \mathrm{L}^{-1}\left(\frac{1}{s}\right) = 1; \;\; g(t) = \mathrm{L}^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{\sin at}{a}.$$

Thus $f(t-u) = 1$; and hence

$$\mathrm{L}^{-1}\left(\frac{1}{s(s^2+a^2)}\right) = \int_0^t 1 \cdot \frac{\sin au}{a} du = \int_0^t \frac{\sin au}{a} du$$

$$= -\frac{1}{a^2}\left[\cos au\right]_0^t = \frac{1}{a^2}\left[1 - \cos at\right]$$

***Example* 2** Using convolution property, find $\mathrm{L}^{-1}\left[\frac{1}{s^2(s-a)}\right]$

*Solution*

Let

$$F(s) = \frac{1}{s^2}, \;\; G(s) = \frac{1}{s-a},$$

then

$$f(t) = \mathrm{L}^{-1}\left(\frac{1}{s^2}\right) = t; \quad g(t) = \mathrm{L}^{-1}\left(\frac{1}{s-a}\right) = e^{at}.$$

Thus,

$$f(t-u) = t-u$$

and hence,

$$\begin{aligned}
L^{-1}\left(\frac{1}{s^2(s-a)}\right) &= \int_0^t \underbrace{(t-u)}_{\text{First Function}} \underbrace{e^{au}}_{\text{Derivative of the second function}} du \\
&= \left[ \underbrace{(t-u)}_{\text{First Function}} \underbrace{\frac{e^{au}}{a}}_{\text{Second function}} \right]_0^t \\
&\quad - \int_0^t \underbrace{(0-1)}_{\text{Derivative of the First Function}} \underbrace{\frac{e^{au}}{a}}_{\text{Second function}} du \\
&= -\frac{t}{a} + \frac{1}{a}\int_0^t e^{au} du \\
&= -\frac{t}{a} + \frac{1}{a}\left[\frac{e^{au}}{a}\right]_0^t = -\frac{t}{a} + \frac{1}{a}\left(\frac{e^{at}-1}{a}\right) \\
&= \frac{1}{a^2}\left[e^{at} - at - 1\right].
\end{aligned}$$

***Example* 3** Using convolution property, find $\mathrm{L}^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]$.

*Solution*

$$\mathrm{L}^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \mathrm{L}^{-1}\left[\frac{1}{s}\frac{s}{(s^2+a^2)^2}\right]$$

Let

$$F(s) = \frac{1}{s}, \;\; G(s) = \frac{s}{(s^2+a^2)^2},$$

then

$$f(t) = \mathrm{L}^{-1}\left(\frac{1}{s}\right) = 1; g(t) = \mathrm{L}^{-1}\left(\frac{s}{(s^2+a^2)^2}\right) = \frac{t\sin at}{2a}; f(t-u) = 1\,.$$

Hence

$$\begin{aligned} \mathrm{L}^{-1}\left[\frac{1}{s}\frac{s}{(s^2+a^2)^2}\right] &= \int_0^t \frac{u\ \sin au}{2a}du \\ &= \frac{1}{2a}\left\{\left[\frac{-u\cos au}{a}\right]_0^t + \int_0^t \frac{\cos au}{a}du\right\} \\ &= \frac{1}{2a^3}\left[\sin at - at\cos at\right], \end{aligned}$$

on simplification.

**Partial Fractions: Repeated Complex Factors** $[(s-a)(s-\bar{a})]^2$

In the chapter "Laplace and Inverse Laplace Transforms" we have mentioned the case of *repeated complex factors* while undergoing partial fraction. In that case the partial fractions are of the form

$$\frac{As+B}{[(s-a)(s-\bar{a})]^2} + \frac{Ms+N}{(s-a)(s-\bar{a})}. \tag{21.8}$$

In the following example we have a partial fraction of the form $\frac{K\omega_0}{\left(s^2+\omega_0^2\right)^2}$ [i.e., only first part of the general form (**??**).]

**Solution of Initial Value Problems**

***Example* 4** Solve the initial value problem

$$y'' + {\omega_0}^2 y = K\sin\omega_0 t$$

with $y(0) = 0$ and $y'(0) = 0$,

*Solution*

$$L[y'' + {\omega_0}^2 y] = L[k \sin \omega_0 t]$$

gives $s^2 Y + {\omega_0}^2 Y = \frac{K\omega_0}{(s^2+\omega_0^2)^2}$

Hence

$$Y = \frac{K\omega_0}{(s^2 + {\omega_0}^2)^2}$$

The denominator is a *repeated complex factor* with roots $s = i\omega_0$ and $-i\omega_0$.

We find its inverse of $Y$ by convolution. Taking $F(s) = \frac{\omega_0}{s^2+\omega_0^2}$, we have $f(t) = L^{-1}[F(s)] = \sin \omega_0 t$ and so we obtain

$$\begin{aligned}
y(t) &= L^{-1}(Y) = L^{-1}\left[\frac{K}{\omega_0} \cdot \frac{\omega_0}{s^2 + {\omega_0}^2} \cdot \frac{\omega_0}{s^2 + {\omega_0}^2}\right] \\
&= L^{-1}\left[\frac{K}{\omega_0} F(s)F(s)\right] \\
&= \frac{K}{\omega_0} L^{-1}\left[F(s)F(s)\right], \text{ applying the linearity} \\
&= \frac{K}{\omega_0}(f * f)(t), \text{ applying convolution} \\
&= \frac{K}{\omega_0}\int_0^t f(t-u)f(u)du \\
&= \frac{K}{\omega_0}\int_0^t \sin\, \omega_0(t-u) \sin \omega_0 u\, du \qquad (21.9)
\end{aligned}$$

Now with $C = \omega_0(t-u)$ and $D = \omega_0 u$, the trigonometric identity

$$-2 \sin C \sin D = \cos(C + D) - \cos(C - D)$$

gives

$$\sin\,\omega_0(t-u)\sin\omega_0 u = -\frac{1}{2}\left\{\cos\omega_0 t - \cos(\omega_0 t - 2\omega_0 u)\right\}$$

$$= \frac{1}{2}\left\{\cos(\omega_0 t - 2\omega_0 u) - \cos\omega_0 t\right\}$$

$= \frac{1}{2}\left\{\cos(2\omega_0 u - \omega_0 t) - \cos\omega_0 t\right\}$, since

$$\cos(\omega_0 t - 2\omega_0 u) = \cos[-(\omega_0 t - 2\omega_0 u)]$$

Using this (21.9) yields

$$\begin{aligned} y(t) &= \frac{K}{2\omega_0}\int_0^t \left\{\cos(2\omega_0 u - \omega_0 t) - \cos\omega_0 t\right\} du \\ &= \frac{K}{2\omega_0}\int_0^{\omega_0 t}\left\{\cos w \frac{1}{2\omega_0} dw\right\} - \cos\omega_0 t\int_0^t du, \end{aligned}$$

where we have taken

$w = 2\omega_0 u - \omega_0 t$; $w = -\omega_0 t$ when $u = 0$;$w = \omega_0 t$ when $u = t$

So

$$\begin{aligned} y(t) &= \frac{K}{2\omega_0}\left(\frac{\sin\,\omega_0 t}{2\omega_0} - \frac{\sin\,(-\omega_0 t)}{2\omega_0} - t\cos\omega_0 t\right) \\ &= \frac{K}{2\omega_0}\left(\frac{\sin\,\omega_0 t}{\omega_0} - t\cos\omega_0 t\right) = \frac{K}{2\omega_0^2}\left(\sin\,\omega_0 t - \omega_0 t\cos\omega_0 t\right). \end{aligned}$$

***Example* 5** Find the solution of the initial value problem

$$y'' + 4y = g(t), \tag{21.10}$$

$$y(0) = 3, \quad y'(0) = -1. \tag{21.11}$$

*Solution*

By taking the Laplace transform of the differential equation and using the initial conditions, we obtain

$$s^2 Y(s) - 3s + 1 + 4Y(s) = G(s),$$

where $Y(s) = L[y(t)]$ and $G(s) = L[g(t)]$.

Hence

$$Y(s) = \frac{3s-1}{s^2+4} + \frac{G(s)}{s^2+4}. \tag{21.12}$$

i.e.,

$$Y(s) = 3\frac{s}{s^2+4} - \frac{1}{2}\frac{2}{s^2+4} + \frac{1}{2}\frac{2}{s^2+4}G(s) \tag{21.13}$$

Then, we obtain

$$y(t) = L^{-1}[Y(s)] = 3\cos 2t - \tfrac{1}{2}\sin 2t + \tfrac{1}{2}\int_0^t \sin 2(t-u)\, g(u)\, du \tag{21.14}$$

If a specific forcing function $g$ is given, then the integral in Eq.(**??**) can be evaluated (by numerical means, if necessary).

**Solution, Transfer Function**

If we denote $\mathrm{L}\left[y(t)\right] = Y(s)$. Then we have

$$\mathrm{L}\left[y'(t)\right] = sY(s) - y(0)$$

and

$$\mathrm{L}\left[y''(t)\right] = s^2 Y(s) - sy(0) - y'(0). \qquad (21.15)$$

Hence applying Laplace transform on the differential equation

$$ay'' + by' + cy = g(t) \qquad (21.16)$$

after simplification, we obtain

$$(as^2 + bs + c)Y = (as + b)y(0) + ay'(0) + L[g(t)].$$

Hence

$$Y(s) = [(as + b)y(0) + ay'(0)]\, H(s) + G(s)\, H(s) \qquad (21.17)$$

with

$$G(s) = \mathrm{L}[g(t)]$$

and

$$H(s) = \frac{1}{as^2 + bs + c}. \qquad (21.18)$$

$H(s)$ is called the **transfer function.**

We write

$$Y(s) = \Phi(s) + \Psi(s)$$

where

$$\Phi(s) = \frac{(as + b)\, y(0) + ay'(0)}{as^2 + bs + c}$$

and

$$\Psi(s) = \frac{G(s)}{as^2 + bs + c}.$$

Then

$$y(t) = \phi(t) + \psi(t)$$

where

$$\phi(t) = L^{-1}[\Phi(s)]$$

and

$$\psi(t) = L^{-1}[\Psi(s)].$$

**Special Case 1:**

If $g(t) = 0$ in (21.16), we have

$$ay'' + by' + cy = 0.$$

This together with the initial conditions $y = (0) = y_0$ and $y'(0) = y'_0$ give the solution

$$y(t) = \phi(t) = L^{-1}\left[\frac{(as + b)\,y_0 + ay'_0}{as^2 + bs + c}\right].$$

**Special case 2:**

If (21.16) is together with the initial conditions $y(0) = 0$ and$y'(0) =$

0, then solution is

$$\begin{aligned} y(t) &= \psi(t) = L^{-1}\left[\frac{G(s)}{ab^2+bs+c}\right] = L^{-1}[H(s)G(s)] \\ &= \int_0^t h(t-u)g(u)du \end{aligned} \tag{21.19}$$

**Special Case 2a:**

If (21.16) is together with the initial conditions $y(0) = 0$ and $y'(0) = 0$, and when $G(s) = 1$, then

$$g(t) = L^{-1}[G(s)] = L^{-1}[1]$$

$= \delta(t)$, the Dirac-delta function, since $L[\delta(t)] = 1$.

Hence

$$\psi(s) = H(s).$$

i.e., $y = h(t)$ is the solution of the initial value problem

$$ay'' + by' + cy = \delta(t), y(0) = 0,\ y'(0) = 0.$$

$h(t)$ is called the **impulse response** of the system.

***Example* 6** Solve the initial value problem

$$y'' + 3y' + 2y = g(t)$$

where $g(t) = \begin{cases} 1, & \text{if } 1 < t < 2 \\ 0, & \text{otherwise,} \end{cases}$ , with the initial conditions $y(0) = y'(0) = 0$.

*Solution*

Comparing with Eq. (21.16) and using Eq. (21.18), we obtain

$$H(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{s+1} - \frac{1}{s+2}$$

and hence

$$h(t) = L^{-1}[H(s)] = L^{-1}\left[\frac{1}{s+1} - \frac{1}{s+2}\right] = e^{-t} - e^{-2t}.$$

Hence

$$y = \psi(t) = L^{-1}[H(s)G(s)] = \int_0^t h(t-\tau)g(\tau)d\tau$$

We are given with $g(t) = 1$ if $1 < t < 2$ and 0 elsewhere. Hence three cases arise.

**Case 1**: If $t < 1$,

$$y = \psi(t) = \int_0^t h(t-\tau)g(\tau)dt$$

$$= \int_0^t h(t-\tau) \times 0 \times dt = 0.$$

**Case 2:** If $1 < t < 2$, we have to integrate from 1 to $t$. This gives

$$y = \psi(t) = \int_1^t h(t-\tau)g(\tau)d\tau$$

$$= \int_1^t [e^{-(t-\tau)} - e^{-2(t-\tau)}]d\tau = \left[e^{-(t-\tau)} - \frac{1}{2}e^{-2(t-\tau)}\right]_1^t$$

$$= e^{-0} - e^{-(t-1)} - \frac{1}{2}(e^{-0} - e^{-2(t-1)}) = \frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)}$$

**Case 3:** If $t > 2$, we have to integrate from 1 to 2 only (not to $t$). This gives

$$y = \psi(t) = \int_1^2 h(t-\tau)g(\tau)d\tau$$

$$= \int_1^2 [e^{-(t-\tau)} - e^{-2(t-\tau)}]d\tau = \left[e^{-(t-\tau)} - \frac{1}{2}e^{-2(t-\tau)}\right]_1^2$$

$$= e^{-(t-2)} - e^{-(t-1)} - \frac{1}{2}[e^{-2(t-2)} - e^{-2(t-1)}].$$

### Integral Equations

Convolution helps in solving certain **integral equations,** that is, equations in which the unknown function $y(t)$ appears under the integral (and perhaps also outside of it). We solve an integral equation in which integral is of the form of a convolution.

***Example* 7** Solve the integral equation

$$y(t) = t + \int_0^t y(\tau)\sin{(t-\tau)}\,d\tau$$

*Solution*

**Step 1:** By observation it can be seen that the integral is of the form of a convolution. Hence we write the given equation as

$$y = t + y * \sin\, t. \tag{21.20}$$

**Step 2:** We write $Y = L(y)$. By an application of Convolution Theorem taking $f(t) = y(t)$ and $g(t) = \sin t$, we have

$$\mathrm{L}\,[y * \sin t] = \mathrm{L}\,[(f * g)(t)] = F(s)G(s) = \mathrm{L}\,[y(t)]\,\mathrm{L}\,[\sin t] = Y(s)\frac{1}{s^2+1}.$$

Hence applying Laplace transform, (21.20) becomes

$$Y(s) = \frac{1}{s^2} + Y(s)\frac{1}{s^2+1}.$$

Solving for $Y(s)$, we obtain

$$Y(s) = \frac{s^2+1}{s^4} = \frac{1}{s^2} + \frac{1}{s^4}.$$

**Step 3:** Taking the inverse transform, we obtain

$$Y(t) = t + \frac{1}{6}t^3.$$

***Example* 8** Solve

$$y(t) = t^3 + \int_0^t \sin(t-u)y(u)du$$

*Solution*

Using convolution,

$$L\left[\int_0^t \sin(t-u)y(u)du\right] = L[\sin t]\, L[y(t)].$$

Hence by applying $L$ on the given integral equation, we get

$$L[y(t)] = L[t^3] + L[\sin t]\, L[y(t)].$$

Hence,

$$L[y(t)]\{1 - L[\sin t]\,\} = L[t^3]$$

or

$$L\,[y(t)] = \frac{L\,[t^3]}{1 - L[\sin t]} = \frac{3!/s^4}{1 - 1/(s^2+1)}$$

$$= \frac{3!}{s^4}\left(\frac{s^2+1}{s^2}\right) = \frac{3!}{s^4} + \frac{3!}{s^6},$$

so

$$y(t) = L^{-1}\left[\frac{3!}{s^4} + \frac{3!}{s^6}\right] = t^3 + \frac{1}{20}t^5$$

is the solution of the given integral equation.

Chapter 22

# Two Point Boundary Value Problems

## 22.1 Two Point Boundary Value Problems

Since the general solution to a second order differential equation

$$y'' + p(x)y' + q(x)y = g(x) \tag{22.1}$$

contains *two* arbitrary constants, we need *two* conditions for obtaining a particular solution. Some times the *two* conditions are of the type

$$y(\alpha) = y_0, \qquad y(\beta) = y_1. \tag{22.2}$$

The conditions in (22.2) are called **boundary conditions** since they refer to the boundary points $\alpha$ , $\beta$ of an interval $I$. Eq. (22.1) and conditions (22.2) together constitute what is known as a **boundary value problem**.

If the function $g$ has the value zero for each $x$, and if the boundary values $y_0$ and $y_1$ are also zero, then the problem (22.1) with conditions (22.2) is called **homogeneous.** Otherwise, the problem is **nonhomogeneous.**

**Remark** The solution of a boundary value problem given by equations (22.1) and (22.2) is unique if and only if no nonzero solution $y$ of (22.1) satisfies $y(A) = y(B) = 0$.

***Example* 1** (*Boundary Value Problem)* Solve

$$y'' + 2y = 0, y(0) = 1, \quad y(\pi) = 0.$$

*Solution*

The general solution to the given differential equation is

$$y = c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x.$$

Using the first of boundary conditions, we obtain

$$y(0) = c_1 = 1.$$

The second boundary condition implies that

$$c_1 \cos\sqrt{2}\pi + c_2 \sin\sqrt{2}\pi = 0,$$

so $c_2 = -\cot\sqrt{2}\pi$. Thus the solution of the given boundary value problem is

$$y = \cos\sqrt{2}x - \cot\sqrt{2}\pi \sin\sqrt{2}x.$$

***Example* 2** (*Boundary Value Problem)* Solve

$$y'' + y = 0, y(0) = 1, \quad y(\pi) = a.$$

*Solution*

The general solution is given by

$$y = c_1 \cos x + c_2 \sin x.$$

Using the first of the boundary conditions we have

$$y(0) = c_1 = 1.$$

The second boundary condition gives

$$y(\pi) = -c_1 = a.$$

These two conditions on $c_1$are incompatible if $a \neq -1$, so the problem has no solution in that case. However, if $a = -1$, then both boundary conditions are satisfied provided that $c_1 = 1$, where $c_2$ is still arbitrary.

In this case a solution to the given boundary value problem is

$$y = 3\cos x + c_2 \sin x,$$

where $c_2$ is still arbitrary. This example illustrates that *a homogeneous boundary value problem may have no solution, and also that under special circumstances it may have infinitely many solutions.*

***Example* 3** Solve the boundary value problem

$$y'' + 2y = 0, y(0) = 0, \quad y(\pi) = 0.$$

*Solution*

The general solution is given by

$$y = c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x.$$

The first boundary condition requires that $c_1 = 0$. The second boundary condition gives $c_2 \sin \sqrt{2}\pi = 0$. Since $\sin \sqrt{2}\pi \neq 0$ we must have $c_2 = 0$. Consequently, $y = 0$ for all $x$ is the only solution of the given boundary value problem. This example illustrates that *a homogeneous boundary value problem may have only the trivial solution* $y = 0$.

***Example* 4** Solve the boundary value problem

$$y'' + y = 0, y(0) = 0, \quad y(\pi) = 0.$$

*Solution*

The general solution is given by

$$y = c_1 \cos x + c_2 \sin x.$$

The first boundary condition requires that $c_1 = 0$. The second boundary condition is also satisfied when $c_1 = 0$ regardless the value of $c_2$. Thus the solution of the given boundary value problem is

$$y = c_2 \sin x,$$

where $c_2$ is still arbitrary. This example illustrates that a homogeneous boundary value problem may have infinitely many solutions.

## 22.2 Eigen Value Problems

Consider the boundary value problem

$$y'' + \lambda y = 0, \tag{22.3}$$

with the boundary conditions

$$y(0) = 0, \quad y(\pi) = 0. \tag{22.4}$$

We define eigen values and eigen functions as follows:

**Definition** The values of $\lambda$ for which nontrivial solutions of (22.3) and (22.4) occur are called **eigen values,** and the corresponding nontrivial solutions are called **eigen functions.**

***Example* 5** Discuss whether $\lambda = 1$ is an eigen value of the boundary value problem (22.3) and (22.4). Give some eigen functions. Is $\lambda = 2$ an eigen value?

*Solution*

Observe that the above boundary value problem (22.3) and (22.4) is the same as the problems in Examples 3 and 4 if $\lambda = 2$ and $\lambda = 1$, respectively.

- Referring to Example 4, we note that for $\lambda = 1$ Eqs.(22.3) and (22.4) have the solution

  $$y = c_2 \sin x,$$

  where $c_2$ is an arbitrary constant. This shows that for $\lambda = 1$ Eqs.(22.3) and (22.4) have nontrivial solution. Hence $\lambda = 1$ is an eigen value. Any nonzero multiple of $\sin x$ is an eigen function corresponding the eigen value $\lambda = 1$.

- Referring to Example 3, we note that for $\lambda = 2$ Eqs.(22.3) and (22.4) have only the trivial solution $y = 0$. Hence $\lambda = 2$ is not an eigen value.

***Example* 6** Find all the eigen values and eigen functions of the boundary value problem

$$y'' + \lambda y = 0, \tag{22.5}$$

with the boundary conditions

$$y(0) = 0, \quad y(\pi) = 0. \tag{22.6}$$

*Solution*

**Case 1)** If $\lambda = 0$, then (22.5) becomes

$$y'' = 0.$$

Integrating (with respect to $x$),

$$y' = a,$$

where $a$ is an arbitrary constant. One more integration (with respect to $x$) yields

$$y = y(x) = ax + b,$$

where $b$ is also an arbitrary constant.

By (22.6), $y(0) = 0$, and hence the above implies $b = 0$. Hence the above reduces to

$$y = y(x) = ax.$$

Again by (22.6), $y(\pi) = 0$, hence the above implies $a\pi = 0$ which implies $a = 0$. Hence the above reduces to

$$y = y(x) = 0.$$

Hence we have only trivial solution when $\lambda = 0$. Hence 0 is not an eigen value.

**Case 2)** If $\lambda > 0$, say $\lambda = \mu^2$, then (22.5) becomes

$$y'' + \mu^2 y = 0,$$

and the characteristic equation corresponding to this second order ordinary differential equation is

$$\lambda^2 + \mu^2 = 0$$

i.e., $\lambda^2 = -\mu^2$.

Hence the general solution of is

$$y(x) = A\cos\mu x + B\sin\mu x.$$

From the boundary condition in (22.6), it follows that

$$y(0) = A = 0\,.$$

Hence

$$y(x) = B\sin\mu x.$$

Again by (22.6),

$$y(\pi) = B\sin\mu\pi = 0.$$

Two cases arise: either $B = 0$ or $\sin\mu\pi = 0$.

The case of $B = 0$ leads to $y(x) = 0$ for $0 \le x \le \pi$, which is the trivial solution. As we are seeking for nontrivial solutions, we take $B \neq 0$. Then

$$\sin\mu\pi = 0.$$

Since sine function has the value zero at every integer multiple of

$\pi$, the above implies

$$\mu\pi = n\pi \quad , \quad n = 1,\ 2,\ 3,\ \ldots$$

or

$$\mu = n \quad , \quad n = 1,\ 2,\ 3,\ \ldots.$$

Hence the corresponding values of $\lambda$ are given by

$$\lambda = \mu^2 = n^2 \quad , \quad n = 1,\ 2,\ 3,\ \ldots.$$

Hence

$$\lambda_1 = 1,\ \lambda_2 = 4,\ \lambda_3 = 9,\ \ldots,\ \lambda_n = n^2,\ \ldots$$

are eigen values of the problem. Since the nontrivial solutions are obtained from

$$y(x) = B\sin\mu x = B\sin\sqrt{\lambda}x,$$

eigen functions corresponding to the eigen value $\lambda_n = n^2$ ($n = 1,\ 2,\ 3,\ \ldots$) are multiples of the functions $\sin nx (n = 1,\ 2,\ 3,\ \ldots)$. We will usually choose the multiplicative constant to be 1 and write the eigen functions as

$$y_1(x) = \sin x,\ y_2(x) = \sin 2x,\ y_3(x) = \sin 3x,\ \ldots,\ y_n(x) = \sin nx,\ \ldots,$$

noting that multiples of these functions are also eigen functions.

**Case 3)** If $\lambda < 0$, say $\lambda = -\mu^2$, then (22.5) becomes

$$y'' - \mu^2 y = 0,$$

and the characteristic equation corresponding to this second order ordinary differential equation is

$$\lambda^2 - \mu^2 = 0$$

i.e., $\lambda^2 = \mu^2 \Rightarrow \ \lambda = \pm\mu.$

Hence the general solution is

$$y(x) = Ae^{\mu x} + Be^{-\mu x}.$$

For the convenience of applying boundary conditions, recalling the definitions of hyperbolic sine and cosine functions, we write the above general solution as

$$y(x) = c_1 \cosh \mu x + c_2 \sinh \mu x.$$

By (22.6), $y(0) = 0$, and hence the above implies

$$c_1 = 0.$$

Thus

$$y(x) = c_2 \sinh \mu x.$$

Also by (22.6), $y(\pi) = 0$, and hence the above implies

$$c_2 \sinh \mu\pi = 0.$$

Since $\mu \neq 0$, $\sinh \mu\pi \neq 0$, hence we must have $c_2 = 0$.

Hence

$$y(x) = 0.$$

Hence we have only trivial solution when $\lambda < 0$. Hence any $\lambda <0$ is not an eigen value.

We conclude that the boundary value problem (22.5) and (22.6)has an infinite sequence of positive eigen values

$\lambda_n = n^2$ for $n = 1,\ 2,\ 3,\ \ldots$

and that the corresponding eigen functions are proportional to $\sin nx$. Further, there are no other real eigen values. There remains the possibility that there might be some complex eigen values. We mention (without proving) that the problem (22.3) and (22.4) have no other complex eigen values.

***Example* 7** Find all the eigen values and eigen functions of the boundary value problem

$$y'' + \lambda y = 0, \tag{22.7}$$

with the boundary conditions

$$y(0) = 0, \quad y(L) = 0. \tag{22.8}$$

*Solution*

Proceeding as in the previous example, it can be seen that the boundary value problem (22.7) and (22.8) has an infinite sequence of positive eigen values

$\lambda_n = \frac{n^2\pi^2}{L^2}$ for $n = 1,\ 2,\ 3,\ \ldots$

and that the corresponding eigen functions are given by

$y_n(x) = \sin\frac{n\pi x}{L}$ for $n = 1,\ 2,\ 3,\ \ldots$

**Exercises**

In Exercises 1-3 solve the following boundary value problems (A basis of solution is given in the bracket).

1. $y'' - 16y = 0; \quad y(0) = 3, \ \ y(\frac{1}{4}) = 3e(e^{4x},\, e^{-4x})$

2. $y'' + y' - 2y = 0; \quad y(0) = 0, \ \ y(1) = e - e^{-2}(e^{x},\, e^{-2x})$

3. $y'' - 2y' = 0; \quad y(0) = -1, \ \ y(\frac{1}{2}) = e - 2(1, \ \ e^{2x})$

   In Exercises 4-9 either solve the given boundary value problem or else show that it has no solution.

4. $y'' + y = 0; \quad y(0) = 0, \ \ y(\pi) = 2$

5. $y'' + 2y = 0; \quad y(0) = 1, \ \ y(\pi) = 0$

6. $y'' + 4y = \cos x; \quad y(0) = 0, \ \ y(\pi) = 0$

7. $y'' + 4y = \sin x; \quad y(0) = 0, \ \ y(\pi) = 0$

8. $x^2y'' - 2xy' + 2y = 0; \quad y(1) = -1, \ \ y(2) = 1$

9. $x^2y'' + 3xy' + y = x^2; \quad y(1) = 0, \ \ y(e) = 0$

***Answers***

1. $y = 3e^{4x}$
2. $y = e^{x} - 2e^{-2x}$
3. $y = -2 + e^{2x}$
4. $y = -2\sin x$
5. $y = \frac{1}{\sqrt{2}}(\cot\sqrt{2}\pi\cos\sqrt{2}x + \sin\sqrt{2}x)$
6. No solution
7. $y = c_2 \sin 2x + \frac{1}{3}\sin x$
8. $y = -\frac{5}{2}x + \frac{3}{2}x^2$
9. $y = -\frac{1}{9}x^{-1} + \frac{1}{9}(1 - e^3)x^{-1}\ln x + \frac{1}{9}x^2$

# Chapter 23

# Fourier Series of Periodic Functions

## 23.1 Introduction

Periodic functions frequently occur in Engineering problems. If these functions be represented in terms of simple periodic functions such as sine and cosine series, then it is of great practical importance. The French physicist Joseph Fourier (1768 - 1830) showed in 1807 that an arbitrary periodic function could be expressed as a linear combination of sines and cosines. These linear combinations are called Fourier series.

In this chapter we discuss basic concepts, facts and techniques in connection with Fourier series. In certain sense these series are more universal than Taylor series. Many discontinuous periodic functions of practical interest can be developed in Fourier series but not in Taylor series.

## 23.2 Periodic Functions

**Definitions** A function $f(x)$ is said to be **periodic** if it is defined for all real $x$ and if there is a positive number $L$ such that

$$f(x+L) = f(x)$$

for every real $x$.

The number $L$ is called a **period** of $f$. The smallest of such $L > 0$ (if it exists) is called the **primitive period** or **fundamental period** of $f$.

**Remark** If $f(x+L) = f(x)$ for every real number $x$, then it follows that

$$f(x+nL) = f(x)$$

for every real number $x$ and for any positive integer $n$.

**Examples**

1. The primitive period of sin $x$ is $2\pi$, because

$$\sin(x+2\pi) = \sin x \quad \forall\ x \in \mathrm{R}$$

and $2\pi$ is the smallest such positive number. Similarly, the primitive period of cos $x$ is $2\pi$.

2. The primitive period of sin $2x$ is $\pi$, because

$$\sin 2\,(x+\pi) = \sin(2x+2\pi) = \sin 2x \quad \forall\ x \in \mathrm{R}$$

   and $\pi$ is the smallest such positive number. Similarly, primitive period of cos $2x$ is $\pi$.

3. The primitive period of sin $3x$ is $\frac{2\pi}{3}$.

4. The primitive period of $\sin 2\pi\, x$ is 1, because

$$\sin 2\pi\,(x+1) = \sin(2\pi x+2\pi) = \sin 2\pi\, x \quad \forall\ x \in \mathrm{R}$$

   and 1 is the smallest such positive number. Similarly, the primitive period of $\cos 2\pi\, x$ is 1.

5. The primitive period of $\sin \pi\, x$ and $\cos \pi\, x$ is 2.

**Examples of periodic functions *without* primitive periods**

The following are some examples of periodic functions *without* primitive periods.:

1. $f(x) = k$, for every real $x$, where $k$ is a constant is a periodic function, and any positive real number is a period. But $f$ has no primitive period.

2. $f$ defined by

$$f(x) = \begin{cases} 1, & \text{when } x \text{ is rational} \\ 0, & \text{when } x \text{ is irrational} \end{cases}$$

is a periodic function, any positive *rational number* is a period. But $f$ has no primitive period.

**Attention!** Hereinafter we use the term **period** for **primitive period.** i.e., $L$ is the period of $f(x)$ if $L$ is the smallest positive number for which

$$f(x+L) = f(x) \quad \text{for every real } x.$$

**Exercises**

In Exercises 1-6, determine whether the given function is periodic. If so, find its fundamental period.

1. $\sin 7x$
2. $\cos 4\pi x$
3. $\sinh 2x$
4. $\sin \frac{\pi x}{L}$
5. $\tan \pi x$
6. $x^3$

7. $f(x) = \begin{cases} 2, & 2n-1 \le x < 2n \\ 3, & 2n \le x < 2n+1 \end{cases} \qquad n = 0, \ \pm 1, \ \pm 2, \ \ldots$

8.

$$f(x) = \begin{cases} 1, & 2n-2 \le x < 2n \\ (-1)^n, & 2n \le x < 2n+2 \end{cases} \qquad n = 0, \ \pm 1, \ \pm 2, \ \ldots$$

***Answers***

1. $L = \frac{2\pi}{7}$
2. $L = \frac{1}{2}$
3. Not periodic
4. $L = 2L$
5. $L = 1$
6. Not periodic

7. $L = 2$ 8. $L = 4$

## 23.3 Trigonometric Series

**Definition** A series of the form

$$a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots$$

$$+ a_n \cos nx + b_n \sin nx + \cdots ,$$

where $a_0, \ a_1, \ b_1, \ \ldots \ , a_n, \ b_n, \ \ldots$ are real constants is called a **trigonometric series**. Here $a_n$'s and $b_n$'s are the coefficients of the series.

### Fourier Series of 2L Periodic Functions

The Fourier series of $2L$ periodic function $f(x)$ initially defined over the interval $-L < x < L$ is given by the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \tfrac{n\pi x}{L} + b_n \sin \tfrac{n\pi x}{L} \right) \tag{23.1}$$

where the coefficients are given by the **Euler formulae**[1]

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x) dx \tag{23.2}$$

[1] Determination of Euler coefficients are done at the end section of this chapter.

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\tfrac{n\pi x}{L}\,dx,\ \ n = 1,\ 2,\ 3,\ldots \tag{23.3}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\tfrac{n\pi x}{L}\,dx,\ \ n = 1,\ 2,\ 3,\ \ldots \tag{23.4}$$

**Theorem 1 (The Fourier Convergence Theorem: Convergence and Sum of Fourier Series)**

Suppose that $f$ and $f'$ are piecewise continuous on the interval $-L \le x < L$. Further, suppose that $f$ is defined outside the interval $-L \le x < L$ so that it is periodic with period $2L$. Then $f$ has a Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\tfrac{n\pi x}{L} + b_n\sin\tfrac{n\pi x}{L}\right)\ldots \tag{23.5}$$

whose coefficients are given by Eqs. (23.2)-(23.4). The Fourier series converges to $f(x)$ at all points where $f$ is continuous, and converges to $\frac{f(x-)+f(x+)}{2}$ at all points where $f$ is discontinuous. That is,

1. If $x$ is a point of continuity, then the sum of the series in (23.5) is $f(x)$. i.e., **at the point of continuity**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\tfrac{n\pi x}{L} + b_n\sin\tfrac{n\pi x}{L}\right). \tag{23.6}$$

2. If $x$ is a point of discontinuity, then the sum of the series in (23.6) is the average of left and right hand limits at $x$. i.e., **at the point of discontinuity**

$$\frac{f(x-)+f(x+)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\,\pi\,x}{L} + b_n \sin\frac{n\,\pi\,x}{L}\right). \tag{23.7}$$

***Example* 1** Find the Fourier series for the 2-periodic function

$$f(x) = \begin{cases} 0, & \text{when } -1 < x < 0 \\ 1, & \text{when } 0 < x < 1 \end{cases}$$

*Solution*

Step 1: Here period is $2L = 2$. $\therefore$ $L = 1$. Substituting $L = 1$ in formulae (23.1) to (23.4), we obtain

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\,\pi\,x + b_n \sin n\,\pi\,x) \tag{23.8}$$

$$a_0 = \int_{-1}^{1} f(x)dx \tag{23.9}$$

$$a_n = \int_{-1}^{1} f(x)\cos n\,\pi\,x\,dx,\ \ n = 1,\,2,\,3,\,\ldots \tag{23.10}$$

$$b_n = \int_{-1}^{1} f(x)\sin n\,\pi\,x\,dx, n = 1,\,2,\,3,\,\ldots \tag{23.11}$$

Step 2: Determination Fourier coefficients using (23.9) to (23.11):

$$a_0 = \int_{-1}^{0} 0 \cdot dx + \int_{0}^{1} dx = \int_{0}^{1} dx = 1.$$

$$\begin{aligned} a_n &= \int_{-1}^{0} 0 \cdot \cos n\,\pi\,x\,dx + \int_{0}^{1} 1 \cdot \cos n\,\pi\,x\,dx \\ &= \left[\frac{\sin n\pi x}{n\pi}\right]_0^1 \\ &= 0. \end{aligned}$$

$$b_n = \int_{-1}^{0} 0 \cdot \sin n\,\pi\,x\,dx + \int_{0}^{1} 1 \cdot \sin n\,\pi\,x\,dx$$

$$= \left[-\frac{\cos n\,\pi\,x}{n\,\pi}\right]_0^1 = \frac{1}{n\,\pi}\{-\cos n\,\pi + 1\} = \frac{1}{n\,\pi}\{1 - (-1)^n\}$$

$$\therefore b_n = \begin{cases} 0\ , & \text{when } n \text{ is even} \\ \frac{2}{n\,\pi}\ , & \text{when } n \text{ is odd} \end{cases}$$

Step 3: Substituting the above values in (23.8), we obtain the Fourier series

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left[\frac{\sin \pi x}{1} + \frac{\sin 3\pi x}{3} + \ . \ . \ . \ \right].$$

***Example* 2** Find the Fourier series of the function

$$f(t) = \begin{cases} 0, \text{when} & -2 < t < -1 \\ k, \text{when} & -1 < t < 1 \\ 0, \text{when} & 1 < t < 2 \end{cases}$$

with period 4.

*Solution*

Step 1: Here period is $2L = 4$. $\therefore$ $L = 2$. Substituting $L = 2$ in formulae (23.1) to (2c) and also changing the variable to $t$, we obtain

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\,\pi\,t}{2} + b_n \sin\frac{n\,\pi\,t}{2}\right) \tag{23.12}$$

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(t)dt \tag{23.13}$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(t)\cos\frac{n\,\pi\,t}{2}\,dt,\;\; n = 1,\,2,\,3,\,\ldots \tag{23.14}$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(t)\sin\frac{n\,\pi\,t}{2}\,dt,\;\; n = 1,\,2,\,3,\,\ldots \tag{23.15}$$

Step 2: Determination Fourier coefficients using (23.13) to (23.15):

$$a_0 = \frac{1}{2}\left\{\int_{-2}^{-1} 0\cdot\,dt + \int_{-1}^{1} k\cdot dt\; + \int_{1}^{2} 0\cdot dt\right\}$$

$$= \frac{k}{2}\int_{-1}^{1} dt\; = \frac{k}{2}\,[t]_{-1}^{\,1} = k.$$

$$a_n = \frac{1}{2}\int_{-\,1}^{1} k\,.\,\cos\frac{n\,\pi\,t}{2}\,dt = \frac{k}{2}\int_{-\,1}^{1}\cos\frac{n\,\pi\,t}{2}\,dt$$

$= \frac{k}{2} \times 2\int_{0}^{1} \cos\frac{n\,\pi\,t}{2}\,dt$ , as cosine functions are even and hence

$\int_{-1}^{1} \cos \frac{n\,\pi\,t}{2}\ dt = 2 \int_{0}^{1} \cos \frac{n\,\pi\,t}{2}\ dt.$

$$= k \left[ \frac{\sin \frac{n\,\pi\,t}{2}}{\frac{n\,\pi}{2}} \right] = \frac{2k}{n\,\pi} \sin \frac{n\,\pi}{2}$$

$$a_n = \begin{cases} 0\ , & \text{when } n \text{ is even} \\ \frac{2k}{n\,\pi}\ , & \text{when } n\ = 1\ ,\ 5\ ,\ 9\ , \\ -\frac{2k}{n\,\pi}\ , & \text{when } n\ = 3\ ,\ 7\ ,\ 11\ , \end{cases}$$

$$b_n = \frac{1}{2} \int_{-1}^{1} k\ \sin \frac{n\,\pi\,t}{2}\ dt = \frac{k}{2} \int_{-1}^{1} \sin \frac{n\,\pi\,t}{2}\ dt$$

$= 0$ for every $n$ , since sine functions are odd and hence $\int_{-1}^{1} \sin \frac{n\,\pi\,t}{2}\ dt = 0.$

Step 3: Substituting the above values in (23.12), we obtain the Fourier series

$$f(t) = \frac{k}{2} + \frac{2k}{\pi} \left[ \cos \frac{\pi\,t}{2} - \frac{1}{3} \cos \frac{3\pi\,t}{2} + \frac{1}{5} \cos \frac{5\pi\,t}{2} - \ \ .\ .\ . \ \right]$$

**Exercises Set A**

In each of the following, find the Fourier series of the periodic function $f(t)$, of period $2L$.

1. $f(t) = \begin{cases} 0, & -2\ <\ t\ <\ 0 \\ 1, & 0 <\ t\ <\ 2 \end{cases}$ , $2L = 4.$

2. $f(t) = \begin{cases} 0, & -1\ <\ t\ <\ 0 \\ t, & 0 <\ t\ <\ 1 \end{cases}$ , $2L = 2.$

3. $f(t)=\begin{cases} -1, & -1 < t < 0 \\ 2t, & 0 < t < 1 \end{cases}, \ 2L=2$

4. Represent $f(x)=\begin{cases} 0, & -l \leq x \leq 0 \\ 1, & 0 < x \leq l \end{cases}$ as a Fourier series in the range $[-l,\ l]$.

5. Find the Fourier series of $f(x)=\begin{cases} 0, & -2 \leq x < 0 \\ k, & 0 \leq x < 2 \end{cases}$ given that $f(x)$ has period 4.

6. Find the Fourier series of $f(x)=\begin{cases} 2, & -2 \leq x < 0 \\ x, & 0 \leq x < 2 \end{cases}$ in $(-2,\ 2)$.

7. Find the Fourier series of $f(x)=\begin{cases} -x, & -2 \leq x < 0 \\ x, & 0 \leq x < 2 \end{cases}$ with $f(x+4)=f(x)$.

8. Find the Fourier series of the function

$$f(t)=\begin{cases} 0, \text{when} & -3<t<-1 \\ 1, \text{when} & -1<t<1 \\ 0, \text{when} & 1<t<3 \end{cases}$$

with period 6. In Exercises 9-10 , find the Fourier series for the given function.

9. $f(x)=\begin{cases} 1, & -L \leq x < 0 \\ 0, & 0 \leq x < L \end{cases}; \ f(x+2L)=f(x)$

10. $f(x) = \begin{cases} x+1, & -1 \le x < 0 \\ 1-x, & 0 \le x < 1 \end{cases} \; ; \; f(x+2) = f(x)$

***Answers to Exercises Set A***

1. $\frac{1}{2} + \frac{2}{\pi}\left[\sin\frac{\pi t}{2} + \frac{1}{3}\sin\frac{3\pi t}{2} + \frac{1}{5}\sin\frac{5\pi t}{2} + \;\;.\;\;.\;\;.\;\;\right]$

2. $\frac{1}{4} - \frac{2}{\pi^2}\left[\cos\pi t + \frac{1}{9}\sin 3\pi t + \cdots\right] + \frac{1}{\pi}\left[\sin\pi t - \frac{1}{2}\sin 2\pi t + \cdots\right]$

3. $1 - \frac{4}{\pi^2}\left[\cos\pi t + \frac{1}{9}\cos 3\pi t - \cdots\right] + \frac{2}{\pi}\left[2\sin\pi t - \frac{1}{2}\sin 2\pi t + \ldots\right]$

4. $\frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\sin\frac{2n+1\pi x}{l}}{2n+1}.$

5. $\frac{k}{2} + \frac{2k}{\pi}\left[\sin\frac{\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} + \frac{1}{5}\sin\frac{5\pi x}{2} + \cdots\right]$

6. $\frac{3}{2} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos nx}{n^2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin nx}{n}.$

7. $1 - \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2n-1)\pi x/2}{(2n-1)^2}$

8. $\frac{1}{3} + \sum_{n=1}^{\infty}\frac{2}{n\pi}\sin\frac{n\pi}{3}\cos\frac{n\pi x}{3}$ or $\frac{1}{3} + \frac{\sqrt{3}}{\pi}\sum_{n=1}^{\infty}\cos\frac{n\pi x}{3}$

9. $\frac{1}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin[(2n-1)\pi x/L]}{2n-1}$

10. $\frac{1}{2} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2n-1)\pi x}{(2n-1)^2}$

## Exercises Set B

1. Find the Fourier series of the function $f(t) = t^2$ on $(-l,\ l)$, $L = 2l$.

2. Find the Fourier series of the function $f(t) = t$ on $(-l,\ l)$, $L = 2l$.

3. Find the Fourier series of $f(t) = \begin{cases} -1, & -1 < t < 0 \\ 1, & 0 < t < 1 \end{cases}$, with period $2L = 2$.

4. $f(t) = 1 - t^2, \quad -1 < t < 1$, with period $2L = 2$.

5. If $f(x) = -x, \quad -L < x < L$ and if $f(x+2L) = f(x)$, find a formula for $f(x)$

   (a). in the interval $L < x < 2L$.

   (b). in the interval $-3L < x < -2L$.

6. If $f(x) = \begin{cases} x+1, & -1 < x < 0 \\ x, & 0 < x < 1 \end{cases}$ and if $f(x+2) = f(x)$, find a formula for $f(x)$

   (a). in the interval $1 < x < 2$. (b) in the interval $8 < x < 9$.

***Answers to Exercises Set B***

1. $a_0 = \frac{2l^2}{3}$, $a_n = \frac{4(-1)^n l^2}{n^2\pi^2}$, and the Fourier series is

$$f(t) = \frac{l^2}{3} + \frac{4l^2}{\pi^2}\left[-\cos\pi t + \frac{1}{2^2}\cos 2\pi t - \frac{1}{3^2}\cos 3\pi t + \cdots\right]$$

2. $b_n = -\frac{2(-1)^n l^2}{n\pi}$, and $f(t) = \frac{2l^2}{\pi}\left[\sin\pi t - \frac{\sin 2\pi t}{2} + \cdots\right]$

3. $\frac{4}{\pi}\left[\sin\pi\, t + \frac{1}{3}\sin 3\pi\, t + \frac{1}{5}\sin 5\pi\, t + \frac{1}{7}\sin 7\pi\, t + \ . \ . \ .\right]$

4. $\frac{2}{3} + \frac{4}{\pi^2}\left[\cos\pi\, t - \frac{1}{4}\cos 2\pi\, t + \frac{1}{9}\cos 3\pi\, t - \ . \ . \ .\right]$

5. (a). $f(x) = 2L - x$ in $L < x < 2L$

   (b). $f(x) = -2L - x$ in $-3L < x < -2L$

6. (a). $f(x) = x - 1$ in $1 < x < 2$

   (b). $f(x) = x - 8$ in $8 < x < 9$

## 23.4 Fourier series of $2\pi$ periodic function defined over the interval $[-\pi,\ \pi]$

We now consider the special case when $2L = 2\pi$. That is, we consider the case when the function is $2\pi$ periodic and initially defined over $[-\pi,\ \pi]$.

Suppose $f(x)$ be defined over the interval $[-\pi,\ \pi]$ and be a periodic function with period $2\pi$, so that $f(x)$ is defined over the set of real numbers. Then the **Fourier series** of $f(x)$ is given by the trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{23.16}$$

where the coefficients $a_n$'s and $b_n$'s are determined by the **Euler formulae**:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx \tag{23.17}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\ dx,\ \ n = 1,\ 2,\ 3,\ \ldots \tag{23.18}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx, \ n = 1, 2, 3, \ldots \tag{23.19}$$

The series given in (23.16) is called the **Fourier series corresponding to** $f(x)$ and its coefficients obtained by the Euler formulae $(2a) - (2c)$ are called the **Fourier coefficients of** $f(x)$.

**Theorem 2 (Convergence and Sum of Fourier Series)** If a periodic function $f(x)$ with period $2\pi$ is piecewise continuous in the interval $-\pi \leq x \leq \pi$ and has a left and right hand derivatives at each point of that interval, then the Fourier series (23.16) of $f(x)$ is convergent. Also,

1. If $x$ is a point of continuity, then the sum of the series in (23.16) is $f(x)$. i.e., **at the point of continuity**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{23.20}$$

2. If $x$ is a point of discontinuity, then the sum of the series in (23.20) is the average of left and right hand limits at $x$. i.e., **at the point of discontinuity**

$$\frac{f(x-) + f(x+)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \, . \tag{23.21}$$

***Example* 3** (*Square wave*) Find the Fourier series of $f$ given by

$$f(x) = \begin{cases} -k, & \text{when } -\pi < x < 0 \\ k, & \text{when } 0 < x < \pi \end{cases} \quad \text{and } f(x + 2\pi) = f(x)$$

and hence deduce the Madhav-Gregory series

$$\frac{\pi}{4} = 1 - \frac{1}{3} - \frac{1}{5} - \quad \cdots \quad .$$

*Solution*

The given function is defined over the interval $[-\pi,\ \pi]$. It is also given that $f(x + 2\pi) = f(x)$. i.e., the given function $f(x)$ is of period $2\pi$.

**Step 1:** Determination of Fourier coefficients using Euler formulae (2a)-2(c):

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{0}(-k)\,dx + \frac{1}{\pi}\int_{0}^{\pi} k\ dx$$

$$= \frac{-k}{\pi}\,[x]_{-\pi}^{\,0} + \frac{k}{\pi}\,[x]_0^{\pi} = \frac{-k\pi}{\pi} + \frac{k\pi}{\pi} = 0.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ dx = \frac{1}{\pi}\int_{-\pi}^{0}(-k)\cos nx\ dx + \frac{1}{\pi}\int_{0}^{\pi} k\ \cos nx\ dx$$

$$= \frac{-k}{\pi}\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} + \frac{k}{\pi}\left[\frac{\sin nx}{n}\right]_{0}^{\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ dx = \frac{1}{\pi}\int_{-\pi}^{0}(-k)\sin nx\ dx + \frac{1}{\pi}\int_{0}^{\pi} k\ \sin nx\ dx$$

$$= \frac{-k}{\pi}\left[-\frac{\cos nx}{n}\right]_{-\pi}^{0} + \frac{k}{\pi}\left[-\frac{\cos nx}{n}\right]_{0}^{\pi} = \frac{2k}{n\,\pi}\,(1 - \cos n\pi)$$

$$= \frac{2k}{n\,\pi}\,(1 - (-1)^n)\,.$$

$$\therefore b_n = \begin{cases} 0, & \text{when } n \text{ is even} \\ \frac{4k}{n\pi}, & \text{when } n \text{ is odd} \end{cases}$$

In particular,

$$b_1 = \frac{4k}{\pi}, \;\; b_2 = 0, \;\; b_3 = \frac{4k}{3\pi},, \;\; b_4 = 0, \;\; b_5 = \frac{4k}{5\pi}, \;\; . \;\; . \;\; .$$

**Step 2:** The Fourier series of $f(x)$ is obtained by putting the coefficients obtained in Step 1 in Eq., and it is

$$\frac{4k}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \;\; . \;\; . \;\; . \;\right] \tag{23.22}$$

**Step 3:** Deduction of the Madhav-Gregory series:

Using Theorem 2 at the point of continuity the series in (23.22) converges to $f(x)$. In other words, at the point of continuity the sum of the series (23.22) is $f(x)$. That is, at the point of continuity

$$f(x) = \frac{4k}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \;\; . \;\; . \;\; . \;\right]. \tag{23.23}$$

$f(x)$ is continuous at $x = \frac{\pi}{2}$. Hence, by (23.23)

$$f\left(\frac{\pi}{2}\right) = \frac{4k}{\pi}\left[\sin\frac{\pi}{2} + \frac{\sin\frac{3\pi}{2}}{3} + \frac{\sin\frac{5\pi}{2}}{5} + \;\; . \;\; . \;\; . \;\right] \tag{23.24}$$

By the definition of $f$, $f(\frac{\pi}{2}) = k$. and by substituting

$\sin\frac{\pi}{2} = 1, \; \sin\frac{3\pi}{2} = -1, \; \sin\frac{5\pi}{2} = 1, \quad . \; . \; .$ in (23.24),

$$k = \frac{4k}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \quad . \;\; . \;\; . \;\right]$$

Hence

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \quad . \quad . \quad .$$

***Example* 4** (*Square wave*) Consider the $2\pi$ periodic function in Example 3. Justify Theorem 2 at the point $x = 0$.

*Solution*

The function $f$ defined by

$$f(x) = \begin{cases} -k, & \text{when } -\pi < x < 0 \\ k, & \text{when } 0 < x < \pi \end{cases} \quad \text{and } f(x + 2\pi) = f(x)$$

has a jump discontinuity at $x = 0$. Hence by Theorem 2,

$$\frac{f(0-) + f(0+)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n0 + b_n \sin n0) . \qquad (23.25)$$

By Example 3, $a_0 = 0$, $a_n = 0$ (for $n = 1, 2, \ldots$), and also $\sin n0 = 0$(for $n = 1, 2, ...$). Hence RHS of (23.25) is 0. Also, $f(0-) = \lim_{x \to 0-} f(x) = -k$ and $f(0+) = \lim_{x \to 0+} f(x) = k$ hence LHS of (23.25) also is 0. Hence Theorem 2 is verified.

***Example* 5** Find the Fourier series expansion for $x^2$ in $-\pi \leq x \leq \pi$.

Hence deduce that

1. $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \quad . \quad . \quad . \quad = \frac{\pi^2}{12}$.
2. $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \quad . \quad . \quad . \quad = \frac{\pi^2}{6}$.
3. $1 + \frac{1}{3^2} + \frac{1}{5^2} + \quad . \quad . \quad . \quad = \frac{\pi^2}{8}$.

*Solution* We postpone the solution of this example to the next chapter "Fourier Series Even and Odd Functions."

***Example* 6** Find the Fourier series of the function

$$f(x) = \begin{cases} x + x^2 & -\pi < x < \pi \\ \pi^2 & \text{when } x = \pm\pi \end{cases}$$

Deduce that $1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{\pi^2}{6}$.

*Solution*

**Step 1:** Using the Euler' formulae (2a)-(2b), we obtain

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(x + x^2\right) dx = \frac{2\pi^2}{3}$, on simplification

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\left(x + x^2\right)}_{u} \underbrace{\cos nx}_{v'} \, dx$$

Using integration by parts,

$$\int uv' = uv - \int u'v,$$

and on simplification, we obtain

$$a_n = \begin{cases} \frac{4}{n^2}, & \text{if } n \text{ is even} \\ -\frac{4}{n^2}, & \text{if } n \text{ is odd} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\left(x + x^2\right)}_{u} \underbrace{\sin nx}_{v'} \, dx$$

$$= \begin{cases} -\frac{2}{n}, & \text{if } n \text{ is even} \\ \frac{2}{n}, & \text{if } n \text{ is odd} \end{cases}, \text{ on simplification.}$$

**Step 2:** Hence, the Fourier series of $f(x)$ is

$$\frac{\pi^2}{3} + \sum_{n=1}^{\infty} \left[ \frac{4(-1)^n}{n^2} \cos nx + \frac{2}{n} (-1)^{n+1} \sin nx \right]$$

i.e., the Fourier series is

$$\frac{\pi^2}{3} - 4 \left[ \cos x - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \ . \ . \ . \right]$$

$$+2 \left[ \sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \ . \ . \ . \right] \tag{23.26}$$

**Step 3:** $f(x)$ is discontinuous at $x = \pi$. Using Theorem 2, at the point of discontinuity the series in (23.26) converges to $\frac{f(x-)+f(x+)}{2}$. That is, at the point of discontinuity

$$\frac{f(x-)+f(x+)}{2} = \frac{\pi^2}{3} - 4 \left[ \cos x - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \ . \ . \ . \right]$$

$$+2 \left[ \sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \ . \ . \ . \right] \tag{23.27}$$

By the definition $f(\pi-) = \lim_{x \to \pi^-} (x + x^2) = \pi + \pi^2$ and by the $2\pi$ periodicity of $f$,

$$f(\pi+) = f(-\pi+) = \lim_{x \to -\pi+} (x + x^2) = -\pi + (-\pi)^2 = -\pi + \pi^2.$$

$$\frac{f(\pi-)+f(\pi+)}{2} = \frac{\pi + \pi^2 - \pi + \pi^2}{2} = \pi^2.$$

Hence, putting $x = \pi$, (23.27) gives

$$\pi^2 = \frac{\pi^2}{3} - 4\left[\cos\pi - \frac{\cos 2\pi}{2^2} + \frac{\cos 3\pi}{3^2} - \frac{\cos 4\pi}{4^2} + \ \ldots\right]$$

i.e.,

$$\pi^2 = \frac{\pi^2}{3} - 4\left[-1 - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} - \ \ldots\right]$$

The above simplifies to

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{\pi^2}{6}.$$

***Example* 7** Find $a_0$ and $a_n$ if

$$x + x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

in $-\pi < x < \pi$ with period $2\pi$.

*Solution* done in the previous example.

***Example* 8** Find the Fourier series of the function

$$f(x) = \begin{cases} 0 & -\pi < x \leq 0 \\ \frac{\pi x}{4} & 0 < x < \pi \end{cases} \quad \text{and } f(x + 2\pi) = f(x).$$

*Solution*

**Step 1:** Using the Euler' formulae, we have

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{\pi^2}{8}$, on simplification

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \begin{cases} 0, & \text{if } n \text{ is even} \\ -\frac{1}{2n^2}, & \text{if } n \text{ is odd} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\sin nx\ dx = \begin{cases} -\frac{\pi}{4n}, & \text{if } n \text{ is even} \\ \frac{\pi}{4n}, & \text{if } n \text{ is odd} \end{cases}$$

**Step 2:** The corresponding Fourier series is

$$\frac{\pi^2}{16} - \frac{1}{2}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ \ldots\right]$$

$$+\frac{\pi}{4}\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \ \ldots\right]$$

***Example* 9** Find the Fourier series of $f$ defined by $f(x) = e^x$ in $(-\pi,\ \pi)$ and $f(x+2\pi) = f(x)$.

*Solution*

**Step 1:** $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2\sinh\pi}{\pi}$, on simplification

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\cos nx\,dx\ \ n = 1,\ 2,\ \ldots$

$$= \frac{2(-1)^n \sinh\pi}{\pi(1+n^2)}$$

$$b_n = -\frac{2n(-1)^n \sinh\pi}{\pi(1+n^2)}$$

**Step 2:** The Fourier series is given by

$$\frac{\sinh\pi}{\pi}\left\{1 + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}(\cos nx - n\sin nx)\right\}.$$

**Exercises**

1. Find the Fourier series expansion of

$$f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

and also deduce that

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

2. Find the Fourier series expansion of

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \pi, & 0 < x < \pi \end{cases}.$$

3. Obtain the Fourier series expansion of the following function of period $2\pi$ :

$$f(x) = x - x^2, \ -\pi < x < \pi.$$

Also deduce that

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} + \cdots$$

***Answers***

1. $-\frac{\pi}{4} - \frac{2}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right] + 3\sin x - \frac{\sin 2x}{2} + \frac{3\sin 3x}{3} - \frac{\sin 4x}{4} \cdots$

2. $\frac{\pi}{2} + 2\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right]$

3.

$$x - x^2 = \frac{\pi^2}{3} + 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\right]$$
$$+2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right]$$

## 23.5 Fourier Series of $2\pi$ Periodic Function defined over *any* Interval of Length $2\pi$

Suppose the given $2\pi$ periodic function $f(x)$ be defined over *any* interval of length $2\pi$, say interval of the form $[0,\ 2\pi]$, $[-\frac{\pi}{2},\ \frac{3\pi}{2}]$, $[\frac{\pi}{2},\ \frac{5\pi}{2}]$, $\cdots$ and even $[-\pi,\ \pi]$.

The **Fourier series** of $2\pi$ periodic function $f(x)$ that is defined over the interval $[\lambda,\ \lambda + 2\pi]$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{23.28}$$

where the coefficients $a_n$'s and $b_n$'s are determined by the **Euler formulae**:

$$a_0 = \frac{1}{\pi}\int_{\lambda}^{\lambda+2\pi} f(x)dx \tag{23.29}$$

$$a_n = \frac{1}{\pi}\int_{\lambda}^{\lambda+2\pi} f(x)\cos nx\ dx,\ \ n = 1,\ 2,\ 3,\ \ldots \tag{23.30}$$

$$b_n = \frac{1}{\pi}\int_{\lambda}^{\lambda+2\pi} f(x)\sin nx\ dx,\ n = 1,\ 2,\ 3,\ \ldots \qquad (23.31)$$

***Example* 10** Find the Fourier series of $f$ given by

$f(x) = \begin{cases} 1, & \text{when } -\frac{\pi}{2} < x < \frac{\pi}{2} \\ -1, & \text{when } \frac{\pi}{2} < x < \frac{3\pi}{2} \end{cases}$ and $f(x) = f(x + 2\pi)$.

*Solution*

Step 1: $f(x)$is defined over the interval $(-\frac{\pi}{2},\ \frac{3\pi}{2})$ and since $f(x) = f(x + 2\pi)$, the given function is $2\pi$ periodic.

The Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

where (taking $\lambda = -\frac{\pi}{2}$ in the Euler formulae)

$$a_0 = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} f(x)dx = \frac{1}{\pi}\left\{\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} 1\,dx + \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}}(-1)\ dx\right\}$$

$$= \frac{1}{\pi}(\pi - \pi) = 0.$$

$a_n = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} f(x)\cos nx\ dx\ \ n = 1,\ 2,\ 3\ ,\ \ldots$

$$= \frac{1}{\pi}\left\{\left[\frac{\sin nx}{n}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \left[\frac{\sin nx}{n}\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\right\}$$

$$= \frac{1}{n\pi}\left\{3\sin\frac{n\pi}{2} - \sin\frac{3n\pi}{2}\right\}$$

We note that

$$\sin\tfrac{3n\pi}{2} = \sin\tfrac{4n\pi - n\pi}{2} = \sin(2n\pi - \tfrac{n\pi}{2})$$

$$= \sin 2n\pi\cos\tfrac{n\pi}{2} - \cos 2n\pi \sin\tfrac{n\pi}{2}$$

$$= -\sin\tfrac{n\pi}{2}$$

Hence

$$a_n = \frac{1}{n\pi}\left\{3\sin\frac{n\pi}{2} - \left(-\sin\frac{n\pi}{2}\right)\right\} = \frac{4}{n\pi}\sin\frac{n\pi}{2}.$$

$$= \begin{cases} 0, & \text{when } n \text{ is even} \\ \frac{4}{n\pi}, & \text{when } n = 1,\ 5,\ \ldots \\ -\frac{4}{n\pi}, & \text{when } n = 3,\ 7,\ \ldots \end{cases}$$

$b_n = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} f(x)\sin nx\ dx,\ n = 1,\ 2,\ 3\ ,.\ .\ .$

$$= \frac{1}{\pi}\left\{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin nx\ dx + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (-1)\ \sin nx\ dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[-\frac{\cos nx}{n}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \left[\frac{\cos nx}{n}\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\right\}$$

$$= \frac{1}{n\pi}\left\{-\cos\frac{n\pi}{2} + \cos\frac{-n\pi}{2} + \cos\frac{3n\pi}{2} - \cos\frac{n\pi}{2}\right\}$$

$$= \frac{1}{n\pi}\left\{-\cos\frac{n\pi}{2} + \cos\frac{3n\pi}{2}\right\}$$

We note that

$$\cos\tfrac{3n\pi}{2} = \cos\tfrac{4n\pi - n\pi}{2} = \cos(2n\pi - \tfrac{n\pi}{2})$$

$$= \cos 2n\pi \cos\tfrac{n\pi}{2} + \sin 2n\pi \sin\tfrac{n\pi}{2}$$

$$= \cos\tfrac{n\pi}{2}$$

Hence

$$a_n = \frac{1}{n\pi}\left\{\cos\frac{n\pi}{2} - \cos\frac{n\pi}{2}\right\} = 0.$$

Step 3: The Fourier series of the given function is

$$f(x) = \frac{4}{\pi}\left[\cos x - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} - \ \ . \ . \ . \ \right].$$

***Example*** **11** Find the Fourier series of $f$ given by

$f(x) = \frac{1}{2}(\pi - x)$ when $0 < x < 2\pi$ and $f(x) = f(x + 2\pi)$.

Deduce that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ \ . \ . \ .$$

*Solution*

Step 1: Here $f(x)$ is defined over the interval $(0,\ 2\pi)$ and since $f(x + 2\pi) = f(x)$, the given function is $2\pi$ periodic.

The Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

where (taking $\lambda = 0$ in the Euler formulae)

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \ dx, \quad n = 1, 2, 3, \ldots$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \ dx, \quad n = 1, 2, 3, \ldots$$

Step 2:

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx = \frac{1}{\pi} \int_0^{2\pi} \tfrac{1}{2}(\pi - x) dx$$

$$= -\frac{1}{4\pi} \left[ (\pi - x)^2 \right]_0^{2\pi} = 0.$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \ dx = \frac{1}{2\pi} \int_0^{2\pi} \underbrace{(\pi - x)}_{u} \underbrace{\cos nx}_{v'} \ dx$$

$$= \frac{1}{2\pi} \left\{ \left[ \frac{(\pi - x) \sin nx}{n} \right]_0^{2\pi} + \int_0^{2\pi} \frac{\sin nx}{n} dx \right\}$$

$= 0$, on simplification.

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \ dx = \frac{1}{2\pi} \int_0^{2\pi} \underbrace{(\pi - x)}_{u} \underbrace{\sin nx}_{v'} \ dx$$

$$= \frac{1}{2\pi}\left\{\left[\frac{-(\pi - x)\cos nx}{n}\right]_0^{2\pi} + \int_0^{2\pi} \frac{\cos nx}{n} dx\right\}$$

$$= \frac{1}{n}.$$

Step 3: Fourier series of the given function is

$$\frac{1}{2}(\pi - x) = \sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \ . \ . \ .$$

Putting $x = \frac{\pi}{2}$ in the above, we obtain

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ . \ . \ .$$

***Example* 12** Find the Fourier series of the function

$$f(x) = x \sin x\,, \qquad 0 < x < 2\pi$$

*Solution*

Here $f(x)$ is defined over the interval $[0,\ 2\pi]$ and since $f(x) = f(x + 2\pi)$, the given function is $2\pi$ periodic.

Now

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} \underbrace{x}_{u}\ \underbrace{\sin x}_{v'}\, dx$$

$$= \frac{1}{\pi}\left\{[x(-\cos x)]_0^{2\pi} - \int_0^{2\pi}(-\cos x)dx\right\}$$

$$= \frac{1}{\pi}\left\{2\pi(-\cos 2\pi) + [\sin x]_0^{2\pi}\right\} = \frac{1}{\pi}\{-2\pi\}$$

$$= -2.$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\ dx = \frac{1}{\pi}\int_0^{2\pi} x\ \sin x\ \cos nx\ dx$$
$$= \frac{1}{\pi}\int_0^{2\pi} x\ \left[\frac{\sin(1+n)x + \sin(1-n)x}{2}\right] dx,$$

since $\sin A\cos B = \frac{\sin(A+B)+\sin(A-B)}{2}$.

$$= \frac{1}{2\pi}\left\{\int_0^{2\pi} x\ \sin(1+n)xdx + \int_0^{2\pi} x\ \sin(1-n)xdx\right\}$$

$$= \frac{1}{2\pi}\left\{\left[x\ \cdot \frac{-\cos(1+n)x}{1+n}\right]_0^{2\pi} - \int_0^{2\pi}\frac{-\cos(1+n)x}{1+n}dx\right\}$$

$$+\frac{1}{2\pi}\left\{\left[x\ \cdot \frac{-\cos(1-n)x}{1-n}\right]_0^{2\pi} - \int_0^{2\pi}\frac{-\cos(1-n)x}{1-n}dx\right\}$$

$$= \frac{1}{2\pi}\left\{2\pi\ \cdot\frac{-\cos(1+n)2\pi}{1+n} + \left[\frac{\sin(1+n)x}{(1+n)^2}\right]_0^{2\pi}\right\}$$

$$+\frac{1}{2\pi}\left\{2\pi\ \cdot\frac{-\cos(1-n)2\pi}{1-n} + \left[\frac{\sin(1-n)x}{(1-n)^2}\right]_0^{2\pi}\right\}$$

$$= \frac{1}{2\pi}\left\{\frac{-2\pi}{1+n}\right\} + \frac{1}{2\pi}\left\{\frac{-2\pi}{1-n}\right\}$$

$$= -1\left(\frac{1}{1+n} + \frac{1}{1-n}\right) = -\frac{2}{1-n^2}$$

By a similar computation, we obtain

$$b_n = 0.$$

Step 3: Fourier series of the given function is

$$f(x) = -1 - 2\sum_{1}^{\infty} \frac{\cos nx}{1-n^2}.$$

***Example* 13** Find the Fourier series expansion of

$$f(x) = \begin{cases} \sin x, & 0 < x < \pi \\ 0, & \pi < x < 2\pi \end{cases}.$$

Also evaluate the series

$$\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots$$

*Solution*

Steps 1 and 2 (Details left to the exercise):

$$a_0 = \frac{2}{\pi}.$$

$$a_n = \begin{cases} \frac{2}{\pi}\left[\frac{1}{1-n^2}\right], & \text{when } n \text{ is even} \\ 0, & \text{when } n \text{ is odd} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\ dx = \frac{1}{\pi}\int_0^{\pi} \sin x\ \sin nx\ dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{\cos(1-n)x - \cos(1+n)x}{2}\ dx... \qquad (23.32)$$

$$= \tfrac{1}{2\pi}\left[\tfrac{\sin(1-n)x}{1-n} - \tfrac{\sin(1+n)x}{1+n}\right]_0^{\pi}, \text{ provided } n \neq 1.$$

When $n = 1$, from (23.32), we have

$$b_1 = \frac{1}{\pi}\int_0^{\pi} \frac{\cos 0 - \cos(1+n)x}{2}\, dx$$

$$= \frac{1}{2\pi}\left[x - \frac{\sin(1+n)x}{1+n}\right]_0^{\pi}$$

$= \frac{1}{2}$, if $n = 1$.

$$\therefore b_n = \begin{cases} \frac{1}{2}, & \text{when } n = 1 \\ 0, & \text{when } n \neq 1 \end{cases}$$

Step 3: The Fourier series is

$$f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\left[\frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \cdots\right] \tag{23.33}$$

Step 4: Evaluation of the series:

The given function is continuous at $x = \frac{\pi}{2}$. Putting $x = \frac{\pi}{2}$ in (23.33) and noting that $f(\frac{\pi}{2}) = \sin\frac{\pi}{2}$, we obtain

$$\sin\tfrac{\pi}{2} = \frac{1}{\pi} + \frac{1}{2}\sin\tfrac{\pi}{2} - \frac{2}{\pi}\left[\frac{\cos(2\cdot\frac{\pi}{2})}{1\cdot 3} + \frac{\cos(4\cdot\frac{\pi}{2})}{3\cdot 5} + \cdots\right]$$

i.e.,

$$1 = \frac{1}{\pi} + \frac{1}{2} - \frac{2}{\pi}\left[\frac{-1}{1\cdot 3} + \frac{1}{3\cdot 5} + \cdots\right]$$

i.e.,

$$\frac{2}{\pi}\left[\frac{-1}{1\cdot 3} + \frac{1}{3\cdot 5} + \cdots\right] = \frac{1}{\pi} - \frac{1}{2}$$

i.e.,

$$\frac{2}{\pi}\left[\frac{-1}{1\cdot 3}+\frac{1}{3\cdot 5}+\ \cdots\ \right]=\frac{2-\pi}{2\pi}$$

i.e.,

$$\frac{2}{\pi}\left[\frac{-1}{1\cdot 3}+\frac{1}{3\cdot 5}+\ \cdots\ \right]=\frac{2-\pi}{2\pi}$$

i.e.,

$$\frac{-1}{1\cdot 3}+\frac{1}{3\cdot 5}+\ \cdots\ =\frac{2-\pi}{4\pi}$$

i.e.,

$$\frac{1}{1\cdot 3}-\frac{1}{3\cdot 5}+\ \cdots\ =\frac{\pi-2}{4}.$$

**Exercises**

In Exercises 1-14, find the Fourier coefficients and the Fourier series of the given function.

1. $f(x)=\begin{cases}-1\,, & -\frac{\pi}{2}\ \leq\ x\ <\ 0\\ 1\,, & \text{if } 0\leq x<\frac{\pi}{2}\\ 0, & \text{if} \frac{\pi}{2}\leq x<\frac{3\pi}{2}\end{cases}$ and $f(x+2\pi)=f(x)$.

2. $f(x)=\begin{cases}1, & -\frac{\pi}{2}\leq x<\frac{\pi}{2}\\ 0, & \frac{\pi}{2}\leq x<\frac{3\pi}{2}\end{cases}$ and $f(x+2\pi)=f(x)$. Also deduce that $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+...=\frac{\pi}{4}$.

$$3. f(x)=x,\ \ 0\leq\ x\ <\ 2\pi,\ \ f(x+2\pi)=f(x).$$

4. $f(x)=\begin{cases}x, & -\frac{\pi}{2}\ \leq\ x\ <\ \frac{\pi}{2}\\ \pi-x, & \frac{\pi}{2}\ \leq\ x\ <\frac{3\pi}{2}\end{cases}$ and $f(x+2\pi)=f(x)$.

5. $f(x) = \begin{cases} -x, & \text{when} -\frac{\pi}{2} \le x < \frac{\pi}{2} \\ x, & \text{when } \frac{\pi}{2} \le x < \frac{3\pi}{2} \end{cases}$ and $f(x+2\pi) = f(x)$.

6. $f(x) = \begin{cases} x, & 0 \le x < \pi \\ 2\pi - x, & \pi \le x < 2\pi \end{cases}$ and $f(x) = f(x+2\pi)$.

Deduce that $1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$.

$$7. f(x) = x^2,\ 0 \le x < 2\pi,\ f(x+2\pi) = f(x).$$

8. Find the Fourier series expansion of

$f(x) = \frac{(\pi - x)^2}{4},\ 0 \le x < 2\pi$ with $f(x+2\pi) = f(x)$.

9. $f(x) = \begin{cases} a, & 0 \le x < \pi \\ -a, & \pi \le x < 2\pi \end{cases}$ and $f(x+2\pi) = f(x)$.

10. $f(x) = \begin{cases} x, & 0 < x < \pi \\ 0, & \pi < x < 2\pi \end{cases}$ and $f(x+2\pi) = f(x)$.

11. $f(x) = x(2\pi - x),\ x \in [0,\ 2\pi]$ and $f(x+2\pi) = f(x)$.

12. $f(x) = \begin{cases} x(\pi - x) & \text{if } 0 < x < \pi \\ -\pi(\pi - x) & \text{if } \pi < x < 2\pi \end{cases}$ with $f(x+2\pi) = f(x)$.

Also deduce that $1 - \frac{1}{3^2} + \frac{1}{5^2} - \ \ldots = \frac{\pi^3}{32}$.

13. $f(x) = \begin{cases} x, & 0 < x < \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} < x < \frac{3\pi}{2} \\ \pi - 2\pi, & \frac{3\pi}{2} < x < 2\pi \end{cases}$ with $f(x+2\pi) = f(x)$.

14. $f(x) = \begin{cases} a, & 0 < x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < x < \frac{3\pi}{2} \\ a, & \frac{3\pi}{2} < x < 2\pi \end{cases}$ with $f(x+2\pi) = f(x)$.

15. Show that in the range 0 to $2\pi$, the expansion of $e^x$ with

period $2\pi$ as Fourier series is

$$e^x = \frac{e^{2\pi} - 1}{\pi} \left\{ \frac{1}{2} + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2+1} - \sum_{n=1}^{\infty} \frac{n\ \sin nx}{n^2+1} \right\}.$$

**Answers**

1. $a_0 = 0, \;\; a_n = 0, \;\; b_n = \begin{cases} \frac{2}{n\pi}, \text{ when } n \text{ is odd} \\ 0, \text{ when } n \text{ is a multiple of } 4 \\ \frac{4}{n\pi}, \text{ when } n \text{ is even but not multiple of } 4 \end{cases}$

and the Fourier series is

$$f(x) = \frac{2}{\pi} \left[ \sin x + \frac{2 \sin 2x}{2} + \frac{\sin 3x}{3} + \frac{2 \sin 6x}{6} + \cdots \right].$$

2. The Fourier series is $\frac{1}{2} + \frac{1}{\pi} \left[ 2\cos x + \cos 2x + \frac{2}{3} \cos 3x + \;\; . \; . \; . \right]$

3. $a_0 = 2\pi, \;\; a_n = 0, \;\; b_n = -\frac{2}{n}$ and the Fourier series is

$$f(x) = \pi - 2 \left[ \sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots \right].$$

4. The Fourier series is $\frac{4}{\pi} \left[ \sin x - \frac{1}{9} \sin 3x + \;\; . \; . \; . \right]$

5. $a_n = \begin{cases} 0, \text{ when } n \text{ is even} \\ -\frac{2}{n}, \text{ when } n = 1, \; 5, \; 9, \; \ldots \\ \frac{2}{n}, \text{ when } n = 3, \; 7, \; 11, \; \ldots \end{cases}, \; b_n = \begin{cases} 0, \text{ when } n \text{ is even} \\ -\frac{4}{n^2\pi}, \text{ when } n = 1, \; 5, \; 9, \; \ldots \\ \frac{4}{n^2\pi}, \text{ when } n = 3, \; 7, \; 11, \; \ldots \end{cases}$

$a_0 = \pi$, and the Fourier series is

$$f(x) = \frac{\pi}{2} - 2 \left[ \cos x - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} - \cdots \right] - \frac{4}{\pi} \left[ \sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \cdots \right].$$

6. $a_0 = \pi$, $a_n = \frac{2(-1)^n - 2}{n^2\pi}$, $b_n = 0$, and the Fourier series is $f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right]$.
For the deduction, put $x = 0$.

8. The Fourier series is $\frac{\pi^2}{12}\left[\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \cdots\right]$. For the deduction, put $x = 0$.

9. The Fourier series is $\frac{4a}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right]$.

10. The Fourier series is $\frac{\pi}{4} - \frac{2}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right]$

$$+\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots$$

$$11. f(x) = \frac{2\pi^2}{3} - 4\left[\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \cdots\right]$$

12. $\frac{8}{\pi}\left[\sin x + \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} + \cdots\right]$. For the deduction put $x = \frac{\pi}{2}$.

13. $$\frac{4}{\pi}\left[\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \cdots\right].$$

14. $$\frac{a}{2} + \frac{2a}{\pi}\left[\frac{\cos x}{1} - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} - \ .\ .\ .\right]$$

## 23.6 Determination of Euler coefficients $a_n$'s and $b_n$'s

Before proceeding further we recall some **results**:

1. $\int_{-L}^{L} dx = [x]_{-L}^{L} = 2L$

2. $\int_{-L}^{L} \cos\frac{n\pi x}{L}\, dx = 0$, where $n$ is an integer.

3. $\int_{-L}^{L} \sin \frac{n\,\pi\,x}{L}\, dx = 0,$ where $n$ is an integer.

   In the following $m,\ n$ are integers:

4. $\int_{-L}^{L} \cos \frac{m\,\pi\,x}{L}\ \cos \frac{n\,\pi\,x}{L}\, dx = \begin{cases} 0, & \text{where}\, m \neq n \\ L, & \text{where}\, m = n \end{cases}$

5. $\int_{-L}^{L} \sin \frac{m\,\pi\,x}{L}\ \sin \frac{n\,\pi\,x}{L}\, dx = \begin{cases} 0, & \text{where}\, m \neq n \\ L, & \text{where}\, m = n \end{cases}$

6. $\int_{-L}^{L} \sin \frac{m\,\pi\,x}{L}\ \cos \frac{n\,\pi\,x}{L}\ dx = \ 0,$ for all $m,\ n$

**Proof**

We prove only some results and leaving all other as exercise.

1. When $n$ is an integer,

$$\int_{-L}^{L} \cos \frac{n\,\pi\,x}{L}\, dx = \left[ \frac{\sin \frac{n\,\pi\,x}{L}}{\frac{n\,\pi}{L}} \right]_{-L}^{T} = \frac{L}{n\pi} \frac{\sin n\pi + \sin n\pi}{n} = 0.$$

2.

$$\int_{-L}^{L} \cos \frac{m\,\pi\,x}{L}\ \cos \frac{n\,\pi\,x}{L}\, dx$$

$$= \frac{1}{2} \left( \int_{-L}^{L} \cos\,(m+n)\ \frac{\pi\,x}{L}\, dx + \int_{-L}^{T} \cos\,(m-n)\ \frac{\pi\,x}{L}\ dx \right)$$

$$= \begin{cases} 0, & \text{if}\ m \neq n \\ L, & \text{if}\ m = n \end{cases}$$

### 23.6.1 Evaluation of Euler Coefficients

Now we are ready to evaluate the Fourier coefficients:

**(*i*) Evaluation of $a_0$ :**

Integrating both sides of the series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \tfrac{n\pi x}{L} + b_n \sin \tfrac{n\pi x}{L}\right) \tag{23.34}$$

with respect to $x$ and in the range $[-L,\ L]$,we obtain

$$\int_{-L}^{L} f\,(x)\ dx = \int_{-L}^{T} \left( \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \tfrac{n\pi x}{L} + b_n \sin \tfrac{n\pi x}{L}\right) \right) dx \tag{23.35}$$

If term-by-term integration of the series is allowed (this is justified, for instance, in the case of *uniformly convergent* series of functions we may integrate term by term), then we obtain

$$\int_{-L}^{L} f(x)\,dx =$$

$$\frac{a_0}{2} \int_{-L}^{T} dx + \sum_{n=1}^{\infty} \left( a_n \int_{-L}^{T} \cos \tfrac{n\pi x}{L}\ dx + b_n \int_{-L}^{T} \sin \tfrac{n\pi x}{L} dx \right) \tag{23.36}$$

By the results discussed just above, (23.36) reduces to

$$\int_{-L}^{L} f(x)\,dx = \frac{a_0}{2} \cdot 2\pi$$

and hence $a_0 = \frac{1}{\pi} \int_{-L}^{L} f(x)\,dx$.

**(*ii*) Evaluation of $a_m$ :**

Multiplying (23.34) by $\cos\frac{m\pi x}{L}$ and integrating with respect to $x$ and in the range $[-L,\ L]$,we obtain

$$\int_{-L}^{L} f(x)\cos\tfrac{m\pi x}{L}\,dx =$$

$$\int_{-L}^{T}\left(a_0+\sum_{n\,=\,1}^{\infty}\left(a_n\cos\tfrac{n\pi x}{L}+b_n\sin\tfrac{n\pi x}{L}\right)\right)\cos\tfrac{nm\pi x}{L}\,dx$$

Assuming that the integration and summation can be interchanged, the above becomes

$$\int_{-L}^{L} f(x)\cos\tfrac{m\pi x}{L}dx = a_0\int_{-L}^{T}\cos\tfrac{m\pi x}{L}dx$$

$$+\sum_{n\,=\,1}^{\infty}\left(a_n\int_{-L}^{L}\cos\tfrac{n\pi x}{L}\ \cos\tfrac{m\pi x}{L}dx+b_n\int_{-L}^{T}\sin\tfrac{n\pi x}{L}\ \cos\tfrac{m\pi x}{L}\,dx\right)$$

Again using the results noted, the above reduces to

$$\int_{-L}^{L} f\,(x)\cos\tfrac{m\pi x}{L}\,dx = a_m\int_{-L}^{T}\cos\tfrac{m\pi x}{L}\ \cos\tfrac{m\pi x}{L}\,dx = La_m$$

and hence $a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx.$

***(ii)* Evaluation of** $b_m$ : Multiplying (23.34) by $\sin\frac{m\pi x}{L}$ and integrating with respect to $x$ and in the range $[-L,\ L]$,we obtain

$$\int_{-L}^{L} f(x)\sin\tfrac{m\pi x}{L}\,dx$$

$$= \int_{-L}^{T} \left( a_0 + \sum_{n\,=\,1}^{\infty} \left(a_n \cos \tfrac{n\pi\, x}{L} + b_n \sin \tfrac{n\pi\, x}{L}\right) \right) \sin \tfrac{m\pi\, x}{L}\, dx$$

Assuming that the integration and summation can be interchanged, the above becomes

$$\int_{-L}^{L} f(x) \sin \tfrac{m\pi\, x}{L}\, dx = a_0 \int_{-L}^{T} \sin \tfrac{m\pi\, x}{L} dx$$

$$+ \sum_{n\,=\,1}^{\infty} \left( a_n \int_{-L}^{L} \cos \tfrac{n\pi\, x}{L}\ \sin \tfrac{m\pi\, x}{L}\, dx + b_n \int_{-L}^{T} \sin \tfrac{n\pi\, x}{L}\ \sin \tfrac{m\pi\, x}{L}\, dx \right)$$

Again using the results noted, the above reduces to

$$\int_{-L}^{L} f(x) \sin \tfrac{m\pi\, x}{L}\, dx = b_m \int_{-L}^{T} \sin \tfrac{m\pi\, x}{L}\ \sin \tfrac{m\pi\, x}{L}\, dx = L\, b_m$$

and hence $b_m = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{m\pi\, x}{L}\, dx$.

Writing $n$ in place of $m$, we have the so-called **Euler formulae:**

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x) dx$$

$$a_n = \frac{1}{T} \int_{-L}^{L} f(x) \cos \tfrac{n\,\pi\, x}{L}\, dx, \quad n = 1, 2, 3, \ldots$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \tfrac{n\,\pi\, x}{L}\, dx, \quad n = 1, 2, 3, \ldots$$

Chapter 24

# Fourier Series of Even and Odd Functions

In this chapter we shall see that the tiresome work for the determination of Fourier coefficients can be considerably reduced if the given periodic function is odd or even. We first review the concepts of even and odd functions.

## 24.1 Even and Odd Functions

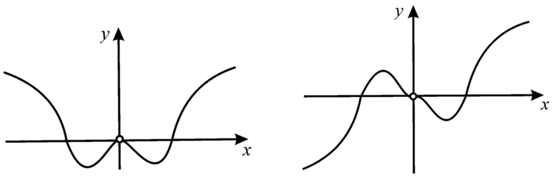


Figure 24.1: Figure on the left side is the graph of an even function. Figure on the right side is the graph of an odd function.

**Definition** A real valued function $f$ is said to be **even function** or simply **even** if

$$f(-x) = f(x) \text{ for all } x.$$

The graph of an even function is symmetrical with respect the $y$-axis.

**Definition** A real valued function $f$ is said to be **odd function** or simply **odd** if

$$f(-x) = -f(x) \text{ for all } x.$$

The graph of such a function is **not** symmetrical with respect the $y$-axis, but symmetrical with respect to the origin.

**Results**

1. The product of two even functions is even.

2. The product of two odd functions is even.

3. The product of an even and an odd function is odd.

*Examples*

Assuming that the given functions are defined over the real line, we have

1. $\cos nx$, $x \sin x$, $\sin x^2$, $\sin^2 x$, $\pi + x^2$, $e^{x^2}$ are all even functions.

2. $\sin nx$, $x \cos x$ are odd functions.

3. $x + x^2$ is neither odd nor even. Similarly, $e^x$ and $\log x$ are neither odd nor even.

4. $f$ defined by $f(x) = k$, where $k$ is a constant is an even function.

5. If $f(x)$ is an even function then $f(x) \cos nx$ is also even.

6. If $f(x)$ is an even function then $f(x) \sin nx$ is odd.

7. If $f(x)$ is an odd function then $f(x) \cos nx$ is odd.

8. If $f(x)$ is an odd function then $f(x) \sin nx$ is even.

**Properties of Even and Odd Functions**

**1.** If $f(x)$ is an **even function**, then

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx.$$

**2.** If $f(x)$ is an **odd function**, then

$$\int_{-a}^{a} f(x)dx = 0.$$

The above properties of even and odd functions can be used to shorten the computational work to find the Fourier series of an even or odd function for the interval $[-L,\ L]$. This is achieved via simplifying the Euler formulae as below:

**1.** If $f(x)$ is an **even function**, then

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx = \frac{2}{L}\int_{0}^{L} f(x)dx.$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\tfrac{n\pi\,x}{L}\ dx\ = \frac{2}{L}\int_{0}^{L} f(x)\cos\tfrac{n\pi\,x}{L}\ dx \quad n\ =\ 1,\ 2,\ 3\ ,\ \ldots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\ \sin\tfrac{n\pi\,x}{L}\ dx = 0 \qquad n\ =\ 1,\ 2,\ 3\ ,\ \ldots$$

**2.** If $f(x)$ is an **odd function**, then

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\ dx\ = 0$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\tfrac{n\pi\,x}{L}\ dx\ = 0 \qquad n\ =\ 1,\ 2,\ 3\ ,\ \ldots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\ \sin\tfrac{n\pi\,x}{L}\ dx = \frac{2}{L}\int_{0}^{L} f(x)\ \sin\tfrac{n\pi\,x}{L}\ dx \qquad n\ =\ 1,\ 2,\ 3\ ,\ \ldots$$

## 24.2 Fourier Cosine Series

**Fourier series for even $2L$ periodic function defined over the interval $[-L,\ L]$**

The Fourier series of an **even** $2L$ periodic function $f(x)$ defined over $[-L,\ L]$ is a **Fourier cosine series** given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \tfrac{n\pi x}{L} \tag{24.1}$$

with coefficients given by

$$a_0 = \frac{2}{L} \int_0^L f(x) dx,\ n = 1,\ 2,\ \ldots \tag{24.2}$$

and

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \tfrac{n\pi x}{L}\, dx,\ n = 1,\ 2,\ \ldots \tag{24.3}$$

***Example* 1** Express the function

$$f(x) = x^2,\ \text{when} \quad -1 < x < 1$$

as a Fourier series with period 2.

*Solution*

Step 1: Here the period $2L = 2$. Hence $L = 1$. Also, since

$$f(-x) = (-x)^2 = x^2 = f(x),$$

the given function is even and hence its Fourier series is the Fourier cosine series given by [Ref. (24.1), (24.2) and (24.3)]:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi\, x \ , \tag{24.4}$$

with coefficients given by

$$a_0 = 2\int_0^1 f(x)dx \tag{24.5}$$

and

$$a_n = 2\int_0^1 f(x)\cos n\pi\, x\, dx, \ \ n = 1,\, 2,\, \ldots \tag{24.6}$$

Step 2:

$$a_0 = 2\int_0^1 x^2\ dx = \ \frac{2}{3}.$$

$$a_n = 2\int_0^1 \underbrace{x^2}_{u}\ \underbrace{\cos n\pi\, x}_{v'}\ dx$$

Applying integration by parts and on simplification, we obtain

$$a_n = \frac{4(-1)^n}{n^2\pi^2}.$$

Step 3: The Fourier series is

$$f(x) = \frac{1}{3} + \frac{4}{\pi^2}\left[-\cos \pi x + \frac{1}{2^2}\cos 2\pi x - \frac{1}{3^2}\cos 3\pi x + \ \cdots\right]$$

***Example 2*** If $f(x)$ is a periodic function with period $2l$, find the Fourier series of $f(x) = x^2$, when $\ -l < x < l$.

*Solution* Proceeding as in the previous example, we obtain

$$f(x) = \frac{l^2}{3} + \frac{4l^2}{\pi^2}\left[-\cos\frac{\pi x}{l} + \frac{1}{2^2}\cos\frac{2\pi x}{l} - \frac{1}{3^2}\cos\frac{3\pi x}{l}\right]$$

## 24.3 Fourier Sine Series

**Fourier series for odd $2L$ periodic function defined over the interval $[-L,\ L]$**

The Fourier series of an **odd** $2L$ periodic function $f(x)$ defined over $[-L,\ L]$ is a **Fourier sine series** given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\tfrac{n\pi x}{L} \tag{24.7}$$

with coefficients

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\tfrac{n\pi x}{L}\,dx,\ \ n = 1,\ 2,\ \ldots \tag{24.8}$$

***Example* 3** Express the function

$$f(x) = x,\ \text{when}\quad -1 < x < 1$$

as a Fourier series with period 2.

*Solution*

Step 1: Here the period $2L = 2$. Hence $L = 1$. Also, since

$$f(-x) = -x = -f(x),$$

the given function is odd and hence its Fourier series is the Fourier sine series given by [Ref. (24.7), and (24.8)]:

$$f(x) = \sum_{n=1}^{\infty} a_n \sin n\pi\, x\ , \tag{24.9}$$

with coefficients given by

$$b_n = 2\int_0^1 f(x) \sin n\pi\, x\, dx,\ \ n = 1,\ 2,\ \ldots \tag{24.10}$$

Step 2:

$$b_n = 2\int_0^1 \underbrace{x}_{u}\ \underbrace{\sin n\pi\, x}_{v'}\ dx$$

Applying integration by parts and on simplification

$$b_n = -\frac{2}{n\pi}(-1)^n,$$

Step 3: The required Fourier series is

$$f(x) = \frac{2}{\pi}\left[\sin \pi x - \frac{\sin 2\pi x}{2} + \ \cdots\ \right]$$

**Exercises Set A**

In each of the following, find the Fourier series of the periodic function $f(t)$, of period $2L$.

1. $f(t) = \begin{cases} 0, & -2 < t < 0 \\ 1, & 0 < t < 2 \end{cases}$, $2L = 4$.

2. $f(t) = \begin{cases} 0, & -1 < t < 0 \\ t, & 0 < t < 1 \end{cases}$, $2\ L = 2$.

3. $f(t) = \begin{cases} -1, & -1 < t < 0 \\ 2t, & 0 < t < 1 \end{cases}$, $2\ L = 2$

4. Represent $f(x) = \begin{cases} 0, & -l \le x \le 0 \\ 1, & 0 < x \le l \end{cases}$ as a Fourier series in the range $[-l,\ l]$.

5. Find the Fourier series of $f(x) = \begin{cases} 0, & -2 \le x < 0 \\ k, & 0 \le x < 2 \end{cases}$ given that $f(x)$ has period 4.

6. Find the Fourier series of $f(x) = \begin{cases} 2, & -2 \le x < 0 \\ x, & 0 \le x < 2 \end{cases}$ in $(-2,\ 2)$..

***Answers to Exercises Set A***

1. $\frac{1}{2} + \frac{2}{\pi}\left[\sin\frac{\pi t}{2} + \frac{1}{3}\sin\frac{3\pi t}{2} + \frac{1}{5}\sin\frac{5\pi t}{2} + \ . \ . \ . \ \right]$

2. $\frac{1}{4} - \frac{2}{\pi^2}\left[\cos\pi t + \frac{1}{9}\sin 3\pi t + \cdots\right] + \frac{1}{\pi}\left[\sin\pi t - \frac{1}{2}\sin 2\pi t + \cdots\right]$

3. $1 - \frac{4}{\pi^2}\left[\cos\pi t + \frac{1}{9}\cos 3\pi t - \cdots\right] + \frac{2}{\pi}\left[2\sin\pi t - \frac{1}{2}\sin 2\pi t + \ldots\right]$

4. $\frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin\frac{2n+1\pi x}{l}}{2n+1}$.

5. $\frac{k}{2} + \frac{2k}{\pi}\left[\sin\frac{\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} + \frac{1}{5}\sin\frac{5\pi x}{2} + \cdots\right]$

6. $\frac{3}{2} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos nx}{n^2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin nx}{n}$.

**Exercises Set B**

1. Find the Fourier series of the function $f(t) = t^2$ on $(-l,\ l)$, L $= 2l$.

2. Find the Fourier series of the function $f(t) = t$ on $(-l,\ l)$, $\mathrm{L} = 2l$.

3. Find the Fourier series of $f(t) = \begin{cases} -1, & -1 < t < 0 \\ 1, & 0 < t < 1 \end{cases}$, with period $2L = 2$.

4. $f(t) = 1 - t^2, \quad -1 < t < 1$, with period $2\mathrm{L} = 2$.

***Answers to Exercises Set B***

1. $a_0 = \frac{2l^2}{3}$, $a_n = \frac{4(-1)^n l^2}{n^2\pi^2}$, and the Fourier series is

$$f(t) = \frac{l^2}{3} + \frac{4l^2}{\pi^2}\left[-\cos\pi t + \frac{1}{2^2}\cos 2\pi t - \frac{1}{3^2}\cos 3\pi t + \cdots\right]$$

2. $b_n = -\frac{2(-1)^n l^2}{n\pi}$, and $f(t) = \frac{2l^2}{\pi}\left[\sin\pi t - \frac{\sin 2\pi t}{2} + \cdots\right]$

$$3. \frac{4}{\pi}\left[\sin\pi t + \frac{1}{3}\sin 3\pi t + \frac{1}{5}\sin 5\pi t + \frac{1}{7}\sin 7\pi t + \ . \ . \ .\right]$$

$$4. \frac{2}{3} + \frac{4}{\pi^2}\left[\cos\pi t - \frac{1}{4}\cos 2\pi t + \frac{1}{9}\cos 3\pi t - \ . \ . \ .\right]$$

## 24.4 Fourier Sine and Cosine Series of $2\pi$ periodic functions

We now consider the special case when functions are $2\pi$ periodic.

**Fourier series for even $2\pi$ periodic function defined over the interval $[-\pi,\ \pi]$**

Suppose $f$ is defined over the interval $[-\pi,\ \pi]$ and is an even $2\pi$ periodic function, then its Fourier series is a cosine series, called

**Fourier cosine series,** and is given by:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \tag{24.11}$$

where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx \tag{24.12}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx \; dx \quad n = 1,\, 2,\, 3 \ldots \tag{24.13}$$

**Attention!** By Theorem 2 in the previous chapter, the equality in (24.11) holds only at the points of continuity.

***Example* 4** Find the Fourier series expansion for $f(x) = x^2$ in $[-\pi,\ \pi]$ with $f(x) = f(x + 2\pi) \quad \forall\ x\ \in$ R.

Hence deduce that

1. $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \ \ . \ . \ . \ \ = \frac{\pi^2}{12}.$
2. $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \ \ . \ . \ . \ \ = \frac{\pi^2}{6}.$
3. $1 + \frac{1}{3^2} + \frac{1}{5^2} + \ \ . \ . \ . \ \ = \frac{\pi^2}{8}.$

*Solution*

**Step 1:** (Verifying that $f$ is even)

Here for $-\pi \le x \le \pi$ , we have by the definition of $f$,

$$f(-x) = (-x)^2 = x^2 = f(x).$$

Hence by the $2\pi$ periodicity of $f$,

$$f(-x) = f(x)$$

for all real values of $x$. Hence $f$ is an even function and hence the Fourier series is the *Fourier cosine series* given by (**??**) whose coefficients are given by (24.12) and (24.13).

**Step 2:** [Determination of Fourier coefficients using (24.12) and (24.13)]

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2\pi^2}{3}.$$

and $a_n = \frac{2}{\pi}\int_0^{\pi} f\,(x)\cos nx\ dx$

$$= \frac{2}{\pi}\int_0^{\pi} \underbrace{x^2}_{u}\underbrace{\cos nx}_{v'}\ dx$$

$$= \frac{2}{\pi}\left(\left[x^2\frac{\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi} 2x\ \frac{\sin nx}{n}\ dx\right),$$

by integration by parts

$$= \frac{2}{\pi}\left(0 - \int_0^{\pi} 2x\ \frac{\sin nx}{n}\ dx\right) = \frac{-4}{n\ \pi}\int_0^{\pi} x\ \sin nx\ dx$$

$$= \frac{-4}{n\,\pi}\left(\left[-x\frac{\cos nx}{n}\right]_0^{\pi} - \int_0^{\pi} -\frac{\cos nx}{n}\ dx\right)$$

$$= \frac{-4}{n\ \pi}\left(-\frac{\pi\ \cos n\pi}{n} + \frac{1}{n}\left[\frac{\sin nx}{n}\right]_0^{\pi}\right) = \frac{4}{n^2}\cos n\pi$$

$$= \frac{4}{n^2}\,(-1)^n\,. = \begin{cases} \frac{4}{n^2}, & \text{if } n \text{ is even} \\ -\frac{4}{n^2}, & \text{if } n \text{ is odd} \end{cases}$$

**Step 3:** Substituting the above values, the Fourier cosine series of the given function is

$$f(x) = \frac{\pi^2}{3} - 4\left[\cos x - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \ \ . \ . \ .\right] \tag{24.14}$$

**Step 4:** (Deduction)

$(i)$ Since $f$ is continuous at $x = 0$, by Theorem 1, equality in (24.14) holds at $x = 0$. Also, and noting that $f(0) = 0^2 = 0$, substituting $x = 0$ in (**??**) we obtain

$$0 = f(0) = \frac{\pi^2}{3} - 4\left[\cos 0 - \frac{\cos 0}{2^2} + \frac{\cos 0}{3^2} - \frac{\cos 0}{4^2} + \ \ . \ . \ .\right] \tag{24.15}$$

i.e., $0 = \frac{\pi^2}{3} - 4\left[1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ \ . \ . \ .\right]$
or $\frac{\pi^2}{3} = 4\left[1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ \ . \ . \ .\right]$
or

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ \cdots . \tag{24.16}$$

$(ii)$ Since $f$ is continuous at $x = \pi$, noting that $f(\pi) = \pi^2$, (**??**) gives

$$\pi^2 = f(\pi) = \frac{\pi^2}{3} - 4\left[\cos \pi - \frac{\cos 2\pi}{2^2} + \frac{\cos 3\pi}{3^2} - \frac{\cos 4\pi}{4^2} + \ \ . \ . \ .\right]$$

i.e.,

$$\pi^2 = \frac{\pi^2}{3} - 4\left[-1 - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} - \ \ . \ . \ .\right]$$

or

$$\frac{2\pi^2}{3} = 4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ \ . \ . \ .\right]$$

or

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ \cdots . \tag{24.17}$$

(*iii*) Adding the series in (24.16) and (24.17), we obtain

$$\frac{\pi^2}{12} + \frac{\pi^2}{6} = 2\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \ \cdots\right]$$

or

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ \cdots .$$

***Example* 5** Find the Fourier series for $f(x) = |x|$ in $[-\pi,\ \pi]$ with $f(x) = f(x + 2\pi) \quad \forall\ x \ \in$ Rand deduce that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}.$$

*Solution*

**Step 1:** (Verifying that $f$ is even)

Here for $-\pi \le x \le \pi$, we have by the definition of $f$,

$$f(-x) = |-x| = |x| = f(x).$$

Hence by the $2\pi$ periodicity of $f$,

$$f(-x) = f(x)$$

for all real values of $x$. Hence $f$ is an even function and hence the Fourier series is the *Fourier cosine series* given by (24.11) whose coefficients are given by (24.12) and (24.13).

**Step 2:** [Determination of Fourier coefficients using (24.12) and (24.13)]

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} |x|\ dx = \frac{2}{\pi}\int_0^{\pi} x\ dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \pi.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\ dx = \frac{2}{\pi}\int_0^{\pi} |x|\ \cos nx\ dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\ \cos nx\ dx = \frac{2}{\pi}\left\{\left[x\frac{\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin nx}{n}\right\}$$

$$= \frac{2}{\pi}\left\{0 - \left[-\frac{\cos nx}{n^2}\right]_0^{\pi}\right\} = \frac{2}{n^2\pi}\left\{(-1)^n - 1\right\}$$

$$= \begin{cases} 0, & \text{if } n \text{ is even} \\ -\frac{4}{n^2\pi}, & \text{if } n \text{ is odd} \end{cases}$$

**Step 3:** Substituting the above values in (24.11), the Fourier cosine series of the given function is

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right]. \qquad (24.18)$$

**Step 4:** (Deduction)

$x = 0$ is a point of continuity hence equality in (24.18) holds at $x = 0$. Noting that $f(0) = 0$, substitution of $x = 0$ in (24.18) yields

$$0 = f(0) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos 0}{1^2} + \frac{\cos 0}{3^2} + \frac{\cos 0}{5^2} + ...\right]$$

or

$$\frac{\pi}{2} = \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ...\right]$$

or

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ... = \frac{\pi^2}{8}.$$

***Example* 6** Find the Fourier series of $f(x) = \begin{cases} -x, & -\pi \leq x < 0 \\ x, & 0 \leq x \leq \pi \end{cases}$ and $f(x + 2\pi) = f(x)$.

*Solution* Note that the given function is nothing but the same in the previous Example.

***Example* 7** Find the Fourier series for $f(x) = x \sin x$ in $[-\pi, \ \pi]$ with $f(x) = f(x + 2\pi) \quad \forall x \in \mathrm{R}$.

*Solution*

**Step 1:** (Verifying that $f$ is even)

Here for $-\pi \leq x \leq \pi$ , we have by the definition of $f$,

$$f(-x) = -x\sin(-x) = -x(-\sin x) = x\sin x = f(x).$$

Hence by the $2\pi$ periodicity of $f$,

$$f(-x) = f(x)$$

for all real values of $x$ (except at multiples of $\pi$). Hence $f$ is an even function and hence the Fourier series is the *Fourier cosine series* given by (24.11) whose coefficients are given by (24.12) and (24.13).

**Step 2:** [Determination of Fourier coefficients using (24.12) and (24.13)]

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin x\ dx$$

$$= \frac{2}{\pi}\left[-x\cos x\right]_0^{2\pi} - \frac{2}{\pi}\int_0^{\pi}(-\cos x)\ dx = \frac{2}{\pi}(-\pi\sin\pi) = 2.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\ dx = \frac{2}{\pi}\int_0^{\pi} x\ \sin x\ \ \cos nx\ dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\ [\sin(n+1)x - \sin(n-1)x]\ \ dx,$$

since $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$$= \frac{1}{\pi}\int_0^{\pi} x\ \sin(n+1)x\ dx - \frac{1}{\pi}\int_0^{\pi} x\ \sin(n-1)x\ dx$$

$$= \begin{cases} -\frac{\cos(n+1)\pi}{n+1}, & \text{when } n = 1 \\ \\ \frac{\cos(n-1)\pi}{n-1} - \frac{\cos(n+1)\pi}{n+1}, & \text{when } n \neq 1 \end{cases}$$

**Step 3:** Substituting the above values, the Fourier cosine series of the given function is

$$f(x) = 1 - \frac{1}{2}\cos x - 2\left[\frac{\cos 2x}{2^2-1} - \frac{\cos 3x}{3^2-1} + \frac{\cos 4x}{4^2-1} - ...\right].$$

## 24.5 Fourier Sine Series

**Fourier series for odd $2\pi$ periodic function defined over the interval $[-\pi,\ \pi]$**

Suppose $f$ is defined over the interval $[-\pi,\ \pi]$ and is an odd $2\pi$ periodic function, then its Fourier series is a sine series, called **Fourier sine series,** and is given by:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \tag{24.19}$$

where

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\ dx \quad (n\ =\ 1,\ 2,\ 3\ ,\ \ldots) \tag{24.20}$$

***Example*** **8** Find the Fourier series of $f$ given by

$$f(x) = x,$$

where $-\pi < x < \pi$ and $f(x) = f(x+2\pi) \quad \forall\ x\ \in$ R.

*Solution*

**Step 1:** (Verifying that $f$ is odd)

Here for $-\pi < x < \pi$, we have by the definition of $f$,

$$f(-x) = -x = -f(x).$$

Hence by the $2\pi$ periodicity of $f$,

$$f(-x) = -f(x)$$

for all real values of $x$ (except at multiples of $\pi$). Hence $f$ is an *odd* function and hence the Fourier series is the *Fourier sine series* given by (24.19) whose coefficients are given by (24.20).

**Step 2:** [Determination of Fourier coefficients using (24.20)]

$$b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\ dx = \frac{2}{\pi}\int_0^\pi x\ \sin nx\ dx$$

$$= \frac{2}{\pi}\left\{\left[x\left(\frac{-\cos nx}{n}\right)\right]_0^\pi - \int_0^\pi \left(-\frac{\cos nx}{n}\right) dx\right\}$$

$$= \tfrac{2}{n\pi}(-\pi\cos n\pi) = \tfrac{2}{n}(-1)^{n+1}, \text{ on simplification}$$

$$= \begin{cases} -\frac{2}{n}, & \text{if } n \text{ is even} \\ \frac{2}{n}, & \text{if } n \text{ is odd} \end{cases}$$

**Step 3:** Substituting the above values in (24.17), the Fourier sine series of the given function is

$$f(x) = 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - ...\right].$$

***Example* 9** Find the Fourier series of $f$ given by

$$f(x) = \sinh ax,$$

where $-\pi < x < \pi$ and $f(x) = f(x + 2\pi) \quad \forall\ x \ \in$ R.

*Solution*

**Step 1:** (Verifying that $f$ is odd)

Here for $-\pi < x < \pi$, we have by the definition of $f$,

$$f(-x) = \sinh(-ax) = -\sinh ax = -f(x).$$

Hence by the $2\pi$ periodicity of $f$,

$$f(-x) = -f(x)$$

for all real values of $x$ (except at multiples of $\pi$). Hence $f$ is an *odd* function and hence the Fourier series is the *Fourier sine series* given by (24.19) whose coefficients are given by (24.20).

**Step 2:** [Determination of Fourier coefficients using (24.20)]

$$b_n = \frac{2}{\pi}\int_0^{\pi}\left(\frac{e^{ax} - e^{-ax}}{2}\right)\sin nx\ dx \tag{24.21}$$

We take $I_1 = \int_0^{\pi} e^{ax}\sin nx\ dx$ and $I_2 = \int_0^{\pi} e^{-ax}\sin nx\ dx$.

Evaluation of $I_1$ :

$$I_1 = \int_0^{\pi} e^{ax}\sin nx\ dx = \left[e^{ax}\left(-\frac{\cos nx}{n}\right)\right]_0^{\pi} - \int_0^{\pi} ae^{ax}\left(-\frac{\cos nx}{n}\right)\ dx$$

$$= \frac{-e^{a\pi}\cos n\pi + 1}{n} + \left[ae^{ax}\left(\frac{\sin nx}{n^2}\right)\right]_0^{\pi} - \int_0^{\pi} a^2 e^{ax}\left(\frac{\sin nx}{n^2}\right)\, dx$$

i.e.,

$$\left(1 + \frac{a^2}{n^2}\right) I_1 = \frac{-e^{a\pi}\cos n\pi + 1}{n} = \frac{e^{a\pi}(-1)^{n+1} + 1}{n}.$$

Hence

$$I_1 = \frac{n}{n^2 + a^2}\left(e^{a\pi}(-1)^{n+1} + 1\right).$$

In a similar manner,

$$I_2 = \frac{n}{n^2 + a^2}\left(e^{-a\pi}(-1)^{n+1} + 1\right).$$

Substituting these values in (A), we obtain

$$b_n = \frac{1}{\pi}(I_1 - I_2) = \frac{1}{\pi}\left\{\frac{(-1)^{n+1}n}{a^2 + n^2}\left(e^{a\pi} - e^{-a\pi}\right)\right\}$$

**Step 3:** Substituting the above values in (24.19), the Fourier sine series of the given function is

$$f(x) = \frac{1}{\pi}\sum\left\{\frac{(-1)^{n+1}n}{a^2 + n^2}\left(e^{a\pi} - e^{-a\pi}\right)\right\}\sin nx.$$

**Exercises**

In Exercises 1 to 6, prove the results.

1. The sum and the product of even functions are even functions.

2. The sum of odd functions is odd. The product of two odd functions is even.

3. The sum of even functions is even. The product of two even functions is even.

4. The product of even and odd functions is odd.

5. If $f(x)$ is odd, then $|f(x)|$ and $f^2(x)$ are even functions.

6. If $f(x)$ is even, then $|f(x)|$ , $f^2(x)$ and $f^3(x)$are even functions.

7. Find the Fourier series of $f(x) = \frac{x^2}{4}, -\pi < x < \pi$ with $f(x + 2\pi) = f(x)$ . Also deduce that

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} + ... = \frac{\pi^2}{12}.$$

8. Find the Fourier series expansion of

$f(x) = \pi^2 - x^2, \quad -\pi < x < \pi$ with $f(x) = f(x + 2\pi)$.

9. Find the Fourier series of $f(x) = \begin{cases} \pi + x, & -\pi < x < 0 \\ \pi - x, & 0 < x < \pi \end{cases}$ and $f(x) = f(x + 2\pi)$.

10. Find the Fourier Coefficients and then Fourier series of $f(x) = \begin{cases} \frac{\pi}{2} + x, & -\pi < x < 0 \\ \frac{\pi}{2} - x, & 0 < x < \pi \end{cases}$ and $f(x + 2\pi) = f(x)$

11. Obtain the Fourier series expansion of the function $f(x) = \begin{cases} 1 + \frac{2x}{\pi}, & -\pi < x < 0 \\ 1 - \frac{2x}{\pi}, & 0 < x < \pi \end{cases}$ of period $2\pi$ : Also deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

12. Obtain the Fourier series expansion of the following function of period $2\pi$ : $f(x) = \begin{cases} -a, & -\pi < x < 0 \\ a, & 0 < x < \pi \end{cases}$ . Also deduce that $\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$.

13. Find the Fourier series expansion of

$$f(x)=\begin{cases} -1+x, & -\pi < x < 0 \\ 1+x, & 0 < x < \pi \end{cases} \quad \text{and } f(x)=f(x+2\pi).$$

Also, deduce that $\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\cdots$

14. Express $\cosh ax$ in Fourier series of period $2\pi$ with $-\pi < x < \pi$.

*15. Find the Fourier series of f given by*

$$f(x)=\begin{cases} -1+x, & \text{when } -\pi < x < 0 \\ 1+x, & \text{when } 0 < x < \pi \end{cases} \quad \text{and } f(x)=f(x+2\pi).$$

***Hints* to Exercises**

9 to 11: If $x$ is such that $0 < x < \pi$, then $-\pi < -x < 0$, and $f(-x)=f(x)$.

12&13: If $x$ is such that $0 < x < \pi$, then $-\pi < -x < 0$, and $f(-x)=-f(x)$.

14: Given is an even function. 15: Given is an odd function.

***Answers***

7. $f(x)=\frac{\pi^2}{12}-\cos x+\frac{1}{4}\cos 2x-\frac{1}{9}\cos 3x+\frac{1}{16}\cos 4x-...$

8. $\frac{2\pi^2}{3}-4\sum_{n=1}^{\infty}\frac{(-1)^n\cos nx}{n^2}$.

9. $\frac{\pi}{2}+\frac{4}{\pi}\left[\cos x+\frac{1}{9}\cos 3x+\frac{1}{25}\cos 5x+\ .\ .\ .\right]$

10.

$$a_0=0,\ a_n=\begin{cases} 0, & \text{when } n \text{ is even} \\ \frac{4}{\pi n^2}, & \text{when } n \text{ is odd} \end{cases}$$

$$f(x)=\frac{4}{\pi}\left[\frac{1}{\cos x}+\frac{1}{9}\cos 3x+\frac{1}{25}\cos 5x+...\right]$$

11. $\frac{8}{\pi^2}\left[\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\ \cdots\right]$

12. $\frac{4a}{\pi}\left[\frac{\sin x}{1}+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+\ \cdots\right]$

13. $b_n = \frac{2}{n\pi} - \frac{2(\pi+1)(-1)^n}{n\pi} = \begin{cases} -\frac{2}{n} & \text{when } n \text{ is even} \\ \frac{2}{n\pi}+\frac{2(\pi+1)}{n\pi} & \text{when } n \text{ is odd} \end{cases}$

and

$f(x) = \frac{2(\pi+2)}{\pi}\sin x - \frac{2}{2}\sin 2x + \frac{2(\pi+2)}{3\pi}\sin 3x - \frac{2}{4}\sin 4x + \frac{2(\pi+2)}{5\pi}\sin 5x -\ \cdots$ For the deduction, put $x = \frac{\pi}{2}$.

14. $f(x) = \frac{1}{a\pi}\sinh a\pi + \frac{a}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}\left[e^{a\pi}-e^{-a\pi}\right]\cos nx.$

15. $f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[1-(1+\pi)(-1)^n\right]\sin nx.$

# Chapter 25

# Half Range Fourier Series - Even and Odd Extensions

## 25.1 Introduction

In this chapter we will see that any function (that need not be periodic) defined over a finite interval $[0,\ L]$ can be represented by means of Fourier series of period $2L$. Before going into the details we define the extension and restriction of functions.

**Definition** Suppose $A \subset B \subset \mathrm{R}$ and suppose that $f : A \to \mathrm{R}$ and $g : B \to \mathrm{R}$ be two functions. We say that the function $g$ is an **extension of the function *f*** if

$$g(x) = f(x) \qquad \forall\ x \in A.$$

In that case we also say that $f$ is the **restriction** of $g$ on $A$.

**Examples** Suppose $A = [0,\ L]$ and $B = [-L,\ L]$ and consider the functions

$$f(x) = x \quad \text{for} \quad x \in [0,\ L] \tag{25.1}$$

and

$$g(x) = x \quad \text{for} \quad x \in [-L,\ L] \tag{25.2}$$

Since $g(x) = f(x) \quad \forall \ x \in [0,\ L]$

$g$ is an extension of the function $f$.

If we also consider the function $h$ defined by

$$h(x) = |x| \quad \text{for} \quad x \in [0,\ L] \tag{25.3}$$

then, since $h(x) = f(x) \quad \forall \ x \in [0,\ L]$

$h$ is also an extension of $f$.

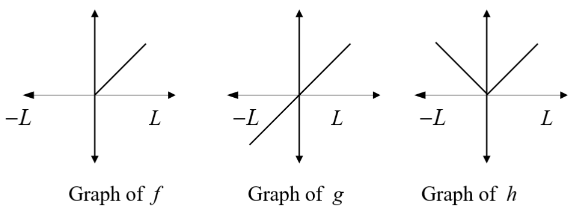


Figure 25.1:

Since $g$ is an odd function, we say $g$ is an **odd extension** of $f$ and since $h$ is an even function, $h$ is an **even extension** of $f$ (Fig.25.1)

Consider the function $f$ considered in Eq. (25.1) above. i.e., $f$ is defined by

$$f(x) = x \quad \forall\, x \in [0,\ L]\,.$$

Clearly, the given function is defined only over an interval of length $L$ and **is not periodic**. But it can be **extended** to an *odd periodic function* or an *even periodic function* as we will discuss now. Depending on the type of extension there are two Fourier series representations for$f(x)$:

(i) when we consider *odd periodic extension function* the corresponding Fourier series would not contain cosine terms. As only sine terms appear in the series and the half part (the cosine part) are missing, the series is called **half range Fourier sine series.**

(i) when we consider *even periodic extension function* the corresponding Fourier series would not contain sine terms. As only cosine appear in the series and the half part (the sine part) are missing, the series is called **half range Fourier cosine series.**

**Half Range Fourier Sine Series**

$g$ defined by (25.2) is an odd extension for $f$ defined by (25.1). If we also suppose that

$$g(x + 2L) = g(x) \text{ for every } x,$$

then $g$ becomes a $2L$ periodic function and hence becomes an **odd $2L$ periodic extension** of the function $f$. Being an odd $2L$ periodic function, the Fourier series for $g$ is the **Fourier sine**

**series** given by

$$g(x) = \sum_{n=1}^{\infty} b_n \sin \tfrac{n\pi\, x}{L},$$

where

$$b_n = \frac{2}{L} \int_0^L g(x) \sin \tfrac{n\pi\, x}{L}\, dx.$$

We consider the above sine series as that one corresponds to the given non-periodic function $f$ and call it the **half range Fourier sine series** for the function $f$ .

**Half Range Fourier Sine Series Formula**

Suppose we are given with a function $f(x)$ defined over the interval $0 \leq x \leq L$ and that $f$ **need not be periodic.** As $g(x) = f(x)$ for $x \in [0,\ L]$, from the last two equations, the **half range Fourier sine series** for the function $f$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \tfrac{n\pi\, x}{L} \tag{25.4}$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \tfrac{n\pi\, x}{L} dx \tag{25.5}$$

**Half Range Fourier Cosine Series**

$h$ defined by (25.3) is an even extension for $f$ defined by (25.1). If we also suppose that

$h(x + 2L) = h(x)$ for every $x$,

then $h$ becomes a $2L$ periodic function and hence becomes an **even $2L$ periodic extension** of the function $f$. Being an even $2L$ periodic function, the Fourier series for $h$ is the **Fourier cosine**

**series** given by

$$h(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \tfrac{n\pi\, x}{L},$$

where

$$a_0 = \frac{2}{L} \int_0^L h(x) dx$$

and

$$a_n = \frac{2}{L} \int_0^L h(x) \cos \tfrac{n\pi\, x}{L}\, dx.$$

We consider the above cosine series as that one corresponds to the given non-periodic function $f$ and call it the **half range Fourier cosine series** for the function $f$ .

**Half Range Fourier Cosine Series Formula**

Suppose we are given with a function $f(x)$ defined over the interval $0 \le x \le L$ and that $f$ **need not be periodic.** As $h(x) = f(x)$ for $x \in [0,\ \pi]$, from the last three equations, the **half range Fourier cosine series** for the function $f$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \tfrac{n\pi\, x}{L} \qquad (25.6)$$

where

$$a_0 = \frac{2}{L} \int_0^L f(x) dx \qquad (25.7)$$

and

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \tfrac{n\pi\, x}{L}\, dx \qquad (25.8)$$

***Example* 1** Find the half range expansions of the function

$$f(t)=\begin{cases}\frac{2k}{l}t, & \text{when} \quad 0\le t<\frac{l}{2}\\ \frac{2k}{l}(l-t), & \text{when} \quad \frac{l}{2}\le t<l\end{cases}$$

*Solution*

Here, we take the given range $0\le t\le l$ as the half range, so that the full range is $-l\le t\le l$ and we take the period as $2L=2l$.

$(i)$ **Half range sine series**

Putting $L=l$ and changing the variable $x$ to $t$, gives

$$f(t)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{l} \tag{25.9}$$

and

$$b_n=\frac{2}{l}\int_0^l f(t)\sin\frac{n\pi t}{l}dt$$

$$=\frac{2}{l}\left\{\int_0^{l/2}\frac{2kt}{l}\sin\frac{n\pi t}{l}dt+\int_{l/2}^{l}\frac{2k}{l}(l-t)\sin\frac{n\pi t}{l}dt\right\}$$

$=\frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}$, on simplification.

*Hence, by* (25.9), *half range sine expansion of* $f(t)$ *is given by*

$$f(t)=\frac{8k}{\pi^2}\left[\frac{1}{1^2}\sin\frac{\pi t}{l}-\frac{1}{3^2}\cos\frac{3\pi t}{l}+\frac{1}{5^2}\cos\frac{5\pi t}{l}-...\right]$$

(ii) **Half range cosine series**

Putting $L = l$ and changing the variable $x$ to $t$, we obtain

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \tfrac{n\pi t}{l} \tag{25.10}$$

and $a_0 = \frac{2}{l} \int_0^l f(t)\, dt$

$$a_n = \frac{2}{l} \int_0^l f(t) \cos \tfrac{n\pi t}{l} dt$$

That is,

$$a_0 = \frac{2}{l} \left\{ \int_0^{l/2} \frac{2kt}{l} dt + \int_{l/2}^{l} \frac{2k}{l}(l-t) dt \right\}$$

$= k$, on simplification

$$a_n = \frac{2}{l} \left\{ \int_0^{l/2} \frac{2kt}{l} \cos \frac{n\pi t}{l} dt + \int_{l/2}^{l} \frac{2k}{l}(l-t) \cos \frac{n\pi t}{l} dt \right\}$$

$$= \frac{4k}{n^2\pi^2} \left( 2\cos\frac{n\pi}{2} - \cos n\pi - 1 \right)$$

Thus $a_2 = -\frac{16k}{2^2\pi^2}$, $\quad a_6 = -\frac{16k}{6^2\pi^2}$, $\quad a_{10} = -\frac{16k}{10^2\pi^2}$, and $a_n = 0$ when $n \neq 2, 6, 10, 14, \ldots$ . Hence, the half range cosine expansion is

$$f(t) = \frac{k}{2} - \frac{16k}{\pi^2} \left[ \frac{1}{2^2} \cos \frac{2\pi t}{l} + \frac{1}{6^2} \cos \frac{6\pi t}{l} + \ldots \right]$$

***Example* 2** Obtain the half range cosine series of

$$f(x) = x, \text{ when } \quad 0 < x < 2.$$

*Solution*

Here, we take the given range $0 \leq x \leq 2$ as the half range, so that the full range is $-2 \leq t \leq 2$ and we take the period as $2T = 2 \times 2 = 4$.

Half range cosine series is obtained as follows:

We have

$$a_0 = \frac{2}{L}\int_0^L f(x)dx = \frac{2}{2}\int_0^2 f(x)dx$$

$$= \int_0^2 x\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^2$$

$$= 2.$$

$$a_n = \frac{2}{T}\int_0^T f(x)\cos\frac{n\pi x}{T}dx = \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx$$

$$= \int_0^2 \underbrace{x}_{u}\ \underbrace{\cos\tfrac{n\pi x}{c}}_{v'}\ dt = \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$= \begin{cases} 0, \text{ when } n \text{ is even} \\ -\frac{8}{n^2\pi^2}, \text{ when } n \text{ is odd} \end{cases}$$

Hence the half range Fourier cosine series is

$$\therefore f(x) = 1 - \frac{8}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \ \cdots\right]$$

***Example* 3** Find the half range sine series of the function

$$f(x) = \begin{cases} x, & \text{when} \quad 0<x<1 \\ 2-x, & \text{when} \quad 1<x<2 \end{cases} \text{ in } 0 \le x \le 2.$$

*Solution*

Here, we take the given range $0 \le x \le 2.$ as the half range, so that the full range is $-2 \le x \le 2.$ and we take the period as $2L = 2 \times 2 = 4$. We have

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx = \frac{2}{2}\int_0^2 f(x)\sin\frac{n\pi x}{2}dx$$

$$= \int_0^1 x \cdot \sin\frac{n\pi x}{2}dx + \int_1^2 (2-x)\sin\frac{n\pi x}{2}dx$$

$= \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}$, on simplification.

*Hence half range sine expansion of $f(x)$ is given by*

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\tfrac{n\pi x}{T} = \sum_{n=1}^{\infty} b_n \sin\tfrac{n\pi x}{2} = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\sin\frac{n\pi}{2}\sin\frac{n\pi\, x}{2}.$$

**Exercises**

1. Represent $f(x) = x, \;\; 0 < x < 1$ by half range (i) Fourier sine series and

   (ii) Fourier cosine series.

2. Expand $f(x) = 2x$ as a series of cosines in $[0, 1]$.

3. Represent $f(x) = \pi - x, \;\; 0 < x < 1$ by a Fourier cosine series.

4. Expand $f(x) = \begin{cases} l, & 0 < x < \frac{l}{2} \\ \frac{2a}{l}(l - x), & \frac{l}{2} < x < l \end{cases}$ as a half range sine series in the interval $0 < x < l$.

5. For the function $f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 4 - 2x, & 1 < x < 2 \end{cases}$ find (i) Fourier cosine series and (ii) Fourier sine series of period 4.

6. Expand $f(x) = x \cos a$ as a half range cosine series in the interval $0 < x < 2$..

7. Represent $f(x) = (x-1)^2, \quad 0 < x < 1$ by Fourier sine series.

8. Express $f(x) = x$ in half range cosine series of periodicity 2l in the range $0 < x < l$.

9. Find the half range Fourier cosine series with period 4 of the function

$$f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$$

10. Find the half range Fourier sine series with period $6\pi$ of the function

$$f(x) = \begin{cases} 0, & 0 < x < \pi \\ 1, & \pi < x < 2\pi \\ 2, & 2\pi < x < 3\pi \end{cases}$$

11. Obtain the half range cosine series for

$$f(t) = \begin{cases} kt, & \text{when} \quad 0 \le t \le \frac{l}{2} \\ k(l-t), & \text{when} \quad \frac{l}{2} \le t \le l \end{cases}$$

12. Obtain the half range cosine series of

$$f(x) = x, \text{ when } \quad 0 < x < l.$$

**Answers**

1. (i) $\frac{1}{2} - \frac{4}{\pi^2}\left[\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \cdots\right]$ (ii) $\frac{2}{\pi}\left[\sin \pi x - \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} - \cdots\right]$

2. $f(x) = \frac{\pi}{2} - \frac{8}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \cdots\right]$

3. (i) $f(x) = \frac{\pi}{2} + \frac{4}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \cdots\right]$

   (ii) $f(x) = 2\left[\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\right]$

4. $\frac{8a}{\pi^2}\left[\sin \frac{\pi x}{l} - \frac{1}{3^2}\sin \frac{3\pi x}{l} + \frac{1}{5^2}\sin \frac{5\pi x}{l} + \cdots\right]$

5. (i) $\sum_{n=1}^{\infty} \frac{8}{n^2\pi^2} \sin \frac{n\pi}{2} \sin \frac{n\pi x}{2}$ (ii) $5 - \frac{16}{\pi^2}\sum_{n=1}^{\infty} \frac{\cos(2n-1)\pi x}{(2n-1)^2}$

6. $\cos a - \frac{8}{n^2\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi x}{2}.$

7. $f(x) = \frac{4}{\pi^3}\sum_{n=1}^{\infty} \frac{\left[(-1)^n + 2n^2\pi^2 - 1\right]}{n^3} \cos n\pi x$

8. $f(x) = \frac{l}{2} - \frac{4l}{\pi^2}\left[\frac{1}{1^2}\cos \frac{\pi x}{2} + \frac{1}{3^2}\cos \frac{3\pi x}{2} + \cdots\right]$

9. $\frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \cos \frac{(2n-1)\pi x}{2}$

10. $\sum_{n=1}^{\infty} \frac{2}{n\pi} \left(\cos \frac{n\pi}{3} + \cos \frac{2n\pi}{3} - 2\cos n\pi\right) \sin \frac{nx}{3}$

11. $f(t) = \frac{kl}{4} - \frac{8kl}{\pi^2} \left[\frac{1}{2^2} \cos \frac{2\pi\, t}{l} + \frac{1}{6^2} \cos \frac{6\pi\, t}{l} + \cdots\right]$

12. $f(x) = \frac{l}{2} - \frac{4l}{\pi^2} \left[\frac{1}{1^2} \cos \frac{\pi x}{2} + \frac{1}{3^2} \cos \frac{3\pi x}{2} + \cdots\right]$

## 25.2 Half Range Series of functions defined over $[0,\ \pi]$

We now consider the special case in which the interval is $[0,\ \pi]$ instead of $[0,\ L]$.

**Half Range Fourier Sine Series Formula**

As $g(x) = f(x)$ for $x \in [0,\ \pi]$, from the last two equations, the **half range Fourier sine series** for the function $f$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \tag{25.11}$$

where

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \ dx \tag{25.12}$$

**Half Range Fourier Cosine Series Formula**

As $h(x) = f(x)$ for $x \in [0,\ \pi]$, from the last three equations, the **half range Fourier cosine series** for the function $f$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \tag{25.13}$$

where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx \tag{25.14}$$

and

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\ dx \tag{25.15}$$

***Example* 4** Obtain the (i) Fourier sine series and (ii) Fourier cosine series for the function

$f(x) = x$ for $x \in [0,\ \pi]$

*Solution*

**(i) Half range Fourier sine series:**

Step 1: If we consider *odd* $2\pi$ *periodic extension*, we get the half range Fourier sine series (25.11) with coefficients given by (25.12).

Step 2: Determination of Fourier coefficients using (25.12).

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\sin nx dx = \frac{2}{\pi}\left[\frac{-x\cos nx}{n}\right]_0^{\pi} + \frac{2}{\pi}\int_0^{\pi}\cos nx dx$$

$$= -\frac{2}{n\pi}(\pi\cos n\pi - 0) + 0 = -\frac{2}{n}\cos n\pi = \frac{2}{n}(-1)^{n+1}$$

$$= \begin{cases} -\frac{2}{n}, & \text{if } n \text{ is even} \\ \frac{2}{n}, & \text{if } n \text{ is odd} \end{cases}$$

Step 3: Substituting these values in (25.11), we obtain the half range Fourier sine series

$$f(x) = 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - ...\right].$$

**(i) Half range Fourier cosine series:**

Step 1: If we consider *even* $2\pi$ *periodic extension*, we get the Fourier half range cosine series given by (25.13) with coefficients given by (25.14) and (25.15).

Step 2: Determination of Fourier coefficients using (25.14) and (25.15).

$$a_0 = \frac{2}{\pi}\int_0^\pi |x|\ dx = \frac{2}{\pi}\int_0^\pi x\ dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^\pi = \pi.$$

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi |x|\ \cos nx\ dx = \frac{2}{\pi}\int_0^\pi x\ \cos nx\ dx \\ &= \frac{2}{\pi}\left\{\left[x\frac{\sin nx}{n}\right]_0^\pi - \int_0^\pi \frac{\sin nx}{n}\right\} \\ &= \frac{2}{\pi}\left\{0 - \left[-\frac{\cos nx}{n^2}\right]_0^\pi\right\} = \frac{2}{n^2\pi}\{(-1)^n - 1\} \\ &= \begin{cases} 0, & \text{if } n \text{ is even} \\ -\frac{4}{n^2\pi}, & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

Step 3: Substituting these values in (25.13), we obtain the half range Fourier cosine series

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right].$$

***Example* 5** Obtain the (i) Fourier sine series and (ii) Fourier cosine series for the function

$$f(x) = \pi - x, \qquad 0 < x < \pi.$$

Assuming the convergence of the series at $x = 0$, deduce that

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ...$$

*Solution*

**(i) Half range Fourier sine series:**

Using (25.11) and (25.12), half range Fourier sine series is obtained as follows:

$$b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx dx = \frac{2}{\pi}\int_0^\pi (\pi - x)\sin nx dx$$

$= \frac{2}{n}$, on simplification.

Hence

$$f(x) = \sum b_n \sin nx = \sum \frac{2}{n}\sin nx = 2\left[\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + ...\right]$$

**(ii) Half range Fourier cosine series:**

Using (25.13), (25.14) and (25.15), half range Fourier cosine series is obtained as follows:

$a_0 = \frac{2}{\pi}\int_0^\pi f(x)dx = \frac{2}{\pi}\int_0^\pi (\pi - x)dx = \pi$, on simplification.

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx \; dx = \frac{2}{\pi}\int_0^\pi \underbrace{(\pi - x)}_{u} \; \underbrace{\cos nx}_{v'} \; dx$$

$$= \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{4}{n^2\pi}, & \text{if } n \text{ is odd} \end{cases}$$

Hence the half range Fourier cosine series for $\pi - x$ in $[0,\ \pi]$ is

$$f(x) = \frac{\pi}{2} + \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + ...\right]. \qquad (25.16)$$

**Deduction of the Series:**

Assuming the convergence of the series at $x = 0$, from (25.16), we obtain

$$\pi - 0 = \frac{\pi}{2} + \frac{4}{\pi}\left[\frac{\cos 0}{1^2} + \frac{\cos 0}{3^2} + \frac{\cos 0}{5^2} + ...\right],$$

which on simplification yields,

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ...$$

***Example* 6** Obtain the Fourier sine series for the function

$$f(x) = c\,, \quad x \in [0,\ \pi].$$

Also deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ \cdots$

*Solution*

Here $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx = \frac{2}{\pi}\int_0^{\pi} c\sin nx dx$

$$= \begin{cases} 0, & \text{when } n \text{ is even} \\ \frac{4c}{n\pi}, & \text{when } n \text{ is odd} \end{cases}$$

Hence the Fourier sine series is

$$f(x) = \frac{4c}{\pi}\left[\sin x + \frac{\sin 3x}{3} + ...\right]$$

i.e.,

$$c = \frac{4c}{\pi}\left[\sin x + \frac{\sin 3x}{3} + ...\right]. \tag{25.17}$$

Putting $x = \frac{\pi}{2}$ in (25.17), we obtain

$$c = \frac{4c}{\pi}\left[\sin\frac{\pi}{2} + \frac{\sin\frac{3\pi}{2}}{3} + ...\right].$$

Hence $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

***Example* 7** Expand $f(x) = \cos x$ in the half range sine-series in $0 \leq x < \pi$.

*Solution*

Here $b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx dx = \frac{2}{\pi}\int_0^\pi \cos x \ \sin nx\, dx$

$$= \begin{cases} \frac{4n}{\pi(n^2-1)}, & \text{when } n \text{ is even} \\ 0, & \text{when } n \text{ is odd} \end{cases}$$

Hence the half range Fourier sine series for the given function is

$$\cos x = \frac{4}{\pi}\left[\frac{2\sin 2x}{2^2-1} + \frac{4\sin 4x}{4^2-1} + \frac{6\sin 6x}{6^2-1} + \cdots\right]$$

**Exercises**

1. Represent $f(x) = x^2, \ \ 0 \leq x < \pi$ by a Fourier sine series.

2. Find the Fourier coefficient and represent $f(x) = 1$, $0 \leq x < \pi$

by Fourier sine series. Hence deduce that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ . \ . \ .$$

3. Show that the Fourier cosine series of $\sin x$ in the half range $0 \le x < \pi$ is $\sin x = \frac{4}{\pi}\left[\frac{1}{2} - \frac{\cos 2x}{1\cdot 3} - \frac{\cos 4x}{3\cdot 5} - \frac{\cos 6x}{5\cdot 7} - \ \cdots\right]$

4. Show that the function $f(x) = \cos 2x, \ \ x \in [0, \ \pi]$ is given by the Fourier sine series $\cos 2x = -\frac{4}{\pi}\left[\frac{\sin x}{3} - \frac{3}{5}\sin 3x - \ \cdots\right]$.

5. Show that the Fourier sine series of $f(x) = \frac{\pi\, x}{8}(\pi - x)$ in the half range $0 \le x < \pi$ is $\frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \ \cdots$

6. Show that the sine series for the function $f(x) = \begin{cases} x, & 0 < x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < x < \pi \end{cases}$ is given by $\frac{2}{\pi}\left[\frac{\sin x}{1^2} + \frac{\pi \sin 2x}{4^2} - \frac{\sin 3x}{3^2} - \frac{\pi \sin 4x}{8} - \ \cdots\right]$.

7. Show that the sine series for the function $f(x) = \begin{cases} 0, & 0 < x < \frac{\pi}{2} \\ c, & \frac{\pi}{2} < x < \pi \end{cases}$ is given by

$$\frac{2c}{\pi}\left[\frac{\sin x}{1} - \frac{2\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} - \frac{2\sin 6x}{6} \ \cdots\right]$$

and the cosine series is given by

$$\frac{2c}{\pi}\left[\frac{\pi}{4} - \frac{\cos x}{1} + \frac{\cos 3x}{3} - \frac{\cos 5x}{5} + \ \cdots\right]$$

8. Show that the cosine series for the function

$$f(x) = \begin{cases} \frac{\pi x}{4}, & 0 < x < \frac{\pi}{2} \\ \frac{\pi}{4}(\pi - x), & \frac{\pi}{2} < x < \pi \end{cases}$$

is given by $\frac{\pi^2}{16} - \frac{1}{2}\left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \ \cdots\right]$

9. Show that $f(x) = \pi x - x^2, \ \ x \in [0, \ \pi]$ can be represented by the Fourier sine series $\frac{8}{\pi}\left[\sin x + \frac{1}{3^3}\sin 3x + \frac{1}{5^3}\sin 5x + \ \cdots\right]$.

10. Show that the Fourier cosine series of the function $f(x) = x\sin x, \ \ x \in [0, \ \pi]$ is given by

$$1 - \frac{\cos x}{2} - \frac{2\cos 2x}{1\cdot 3} + \frac{2\cos 3x}{2\cdot 4}\frac{2\cos 4x}{3\cdot 5} + \ \cdots$$

Hence deduce that $\frac{1}{2} + \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \ \cdots = \frac{\pi}{4}$.

11. Expand $f(x) = \pi x - x^2, \ \ x \in [0, \ \pi]$ in Fourier cosine series.

12. Express $f(x) = x\sin x, \ \ x \in [0, \ \pi]$ in half range sine series.

**Answers**

1. $f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\left\{\frac{\pi^2}{n} + \frac{2}{n^3}\left[(-1)^n - 1\right]\right\}\sin nx$

2. $b_n = \frac{2}{n\,\pi}\left[1 + (-1)^{n+1}\right], \quad f(x) = \frac{4}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \ldots\right]$

4. $f(x) = \frac{4}{\pi}\left[\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - \ \ldots\right]$

11. $\pi\, x - x^2 = \frac{\pi}{6} - 2\sum_{n=1}^{\infty}\frac{1+(-1)^n}{n^2}\cos nx$

12. $f(x) = \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{\cos\pi(1-n)}{(1-n)^2} - \frac{\cos\pi(1+n)}{(1+n)^2}\right]\sin nx$

Chapter 26

# Partial Differential Equations - A Quick Review [1]

## 26.1 Partial Differential Equations

An equation involving partial derivatives is a **partial differential equation.** In this chapter we consider some partial differential equations. We consider the solution of the wave equation also.

Some of the important linear partial differential equations of the second order are given below:

*One dimensional wave equation:*

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{26.1}$$

[1] Introductory Chapter on partial differential equations. Not mentioned in the syllabus.

*One dimensional heat equation:*

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{26.2}$$

*Two dimensional Laplace equation:*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{26.3}$$

*Two dimensional Poisson equation:*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \tag{26.4}$$

*Three dimensional Laplace equation:*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0. \tag{26.5}$$

In the above $c$ is a constant, $t$ is time, and $x$, $y$, $z$ are Cartesian coordinates. Equation (26.4) with $f$ not identically zero is **non-homogeneous,** while the other equations are **homogeneous.**

**Definition** A **solution** of a partial differential equation in some region $R$ is a function which has all the partial derivatives appearing in the equation in some domain containing $R$, and satisfies the equation everywhere in $R$.

***Example* 1** Show that the functions

$$u = x^2 - y^2, \;\; u = e^x \cos y, \;\; u = \log\ (x^2 + y^2)$$

are *solutions* of the two dimensional Laplace equation (26.3).

*Solution*

When $u = x^2 - y^2$, we have

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = -2y, \quad \frac{\partial^2 u}{\partial x^2} = 2 \text{ and } \frac{\partial^2 u}{\partial y^2} = -2.$$

Hence $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ so that $u = x^2 - y^2$ satisfies the two dimensional Laplace equation (26.3). Similarly, it can be verified $u = e^x \cos y$ and $u = \log\ (x^2 + y^2)$ are also solutions of (26.3).

***Example* 2** Verify that $u = x^3 + 3xt^2$ is a solution of the one dimensional wave equation (26.1), for a suitable value of $a$.

*Solution*

When $u = x^3 + 3xt^2$, we obtain

$$\frac{\partial u}{\partial x} = 3x^2 + 3t^2,$$

$$\frac{\partial^2 u}{\partial x^2} = 6x,$$

$$\frac{\partial u}{\partial t} = 6xt \text{ and } \frac{\partial^2 u}{\partial t^2} = 6x.$$

Putting the above values in the one dimensional wave equation (26.1),

$6x = a^2 6x$, for $a = 1$ or for $a = -1$.

Hence $u = x^3 + 3xt^2$ satisfies the one dimensional wave equation when $a = \pm 1$

**Theorem 1 (Fundamental Theorem for Linear Homoge-**

**neous Partial Differential Equations)**

If $u_1$ and $u_2$ are any solutions of a linear homogeneous partial differential equation in some region, then

$$u = c_1 u_1 + c_2 u_2$$

where $c_1$ and $c_2$ are any constants, is also a solution of that equation in that region.

## 26.2 Relation to Ordinary Differential Equation

If a partial differential equation involves derivatives with respect to one of the independent variables only, we can solve it like an ordinary differential equation, treating the other independent variable as constant. This is illustrated through the following examples.

**Notation** $u_x, \; u_{xx}, \; u_y, \; u_{yy}, \; u_{xy}, \; u_{yx}$ denote the partial derivatives

$$\frac{\partial u}{\partial x}, \quad \frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial u}{\partial y}, \quad \frac{\partial^2 u}{\partial y^2}, \quad \frac{\partial^2 u}{\partial x \partial y}, \quad \frac{\partial^2 u}{\partial y \partial x}$$

respectively. More precisely,

$$u_x(x, \; y) = \frac{\partial u}{\partial x}, \;\; u_{xx}(x, \; y) = \frac{\partial^2 u}{\partial x^2}, \;\; u_y(x, \; y) = \frac{\partial u}{\partial y},$$

$$u_{yy}(x, \; y) = \frac{\partial^2 u}{\partial y^2}, \;\; u_{xy}(x, \; y) = \frac{\partial^2 u}{\partial x \partial y}, \;\; u_{yx}(x, \; y) = \frac{\partial^2 u}{\partial y \partial x}$$

Recall that under some conditions

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}.$$

i.e.,

$$u_{xy} = u_{yx}.$$

***Example* 3** Solve the partial differential equation

$$u_{xx} + 4u = 0$$

where $u$ is a function of two variables $x$ and $y$.

*Solution*

Given

$$u_{xx} + 4u = 0,$$

i.e.,

$$\frac{\partial^2 u}{\partial x^2} + 4u = 0.$$

The given partial differential equation involves derivatives with respect to the independent variable $x$ only. We solve the given partial differential equation, considering as an ordinary differential equation in $x$, keeping $y$ as constant. Then characteristic equation is

$$\lambda^2 + 4 = 0$$

and hence

$$\lambda = \pm 2i.$$

Hence the solution to the given equation is given by $u(x, y) = f(y)\cos 2x + g(y)\sin 2x$ where $f(y)$ and $g(y)$ are arbitrary functions of $y$ alone.

***Example* 4** Solve the partial differential equation

$$u_{yy} + 4u = 0$$

where $u$ is a function of two variables $x$ and $y$.

*Solution*

Here the partial differential equation involves only the derivatives in the variable $y$. Proceeding as in Example 3, we obtain

$$u(x, y) = f(x)\cos 2y + g(x)\sin 2y$$

where $f(x)$and $g(x)$are arbitrary functions of x alone.

***Example* 5** Solve the partial differential equation

$$u_{xx} - 4 = 0$$

where $u$ is a function of two variables $x$ and $y$.

*Solution*

The partial differential equation involves derivatives with respect to the variable $x$ only. Hence the given equation can be treated as an ordinary differential equation in $x$ with $y$ as constant.

$$\frac{\partial^2 u}{\partial x^2} - 4 = 0$$

implies

$$\frac{\partial^2 u}{\partial x^2} = 4.$$

Integrating with respect to $x$, we obtain

$$\frac{\partial u}{\partial x} = 4x + f(y),$$

where $f(y)$ is an arbitrary function of $y$ alone.

Integrating again with respect to $x$, we obtain

$$\frac{\partial u}{\partial x} = 2x^2 + xf(y) + g(y).$$

***Example* 6** Solve the partial differential equation $u_y + 2\ y\ u = 0$, where $u$ is a function of two variables $x$ and $y$.

*Solution*

The given partial differential equation involves derivatives with respect to the independent variable $y$ only. We solve the given partial differential equation, considering it as an ordinary differential equation in $y$, keeping $x$ as constant. By separating variables, the given equation is

$$\frac{\partial u}{u} = -2y\,\partial y$$

Integrating,

$$\log u = -y^2 + \log f(x)$$

or

$$u = f(x)e^{-y^2}.$$

***Example* 7** Setting $u_x = p$, solve $u_{xy} = u_x$.

*Solution*

$$u_{xy} = u_x$$

can be written as

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial u}{\partial x}$$

or

$$\frac{\partial}{\partial y}(p) = p.$$

Since $p$ is the dependent variable and $y$ is the only independent variable, the last equation can be considered as the first order ordinary equation

$$\frac{dp}{dy} = p \tag{26.6}$$

regarding $x$ as constant. By separating variables, (26.1) becomes

$$\frac{dp}{p} = dy$$

which on integration with respect to $y$ partially yields

$$\log p = y + \log f(x)$$

where $f(x)$ is an arbitrary function of $x$ alone. Hence

$$p = e^y f(x)$$

i.e.,

$$\frac{\partial u}{\partial x} = e^y f(x).$$

Now considering $y$ as a constant and integrating partially with respect to $x$, we obtain the following as the solution of the given equation

$$u = e^y \int f(x)dx + g(y)$$

where $g(y)$ is an arbitrary function of $y$ alone.

## 26.3 Product Method (Separation of Variables)

We now discuss the product method for finding the solution $u(x,\ y)$ of a given partial differential equation.

***Example* 8** Find the solution of $u_x + u_y = 0$ by separating variables (product method).

The given partial differential equation is

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \ldots \tag{26.7}$$

Let the solution of this equation be

$$u(x,\ y) = X(x)Y(y)$$

where $X(x)$ is a function of $x$ alone and $Y(y)$ is a function of $y$ alone.

Let $X' = \frac{dX}{dx}$ and $\dot{Y} = \frac{dY}{dy}$ , so that (26.7) becomes the form

$$X'Y + X\dot{Y} = 0$$

or

$$X'Y = -X\dot{Y}.$$

or

$$\frac{X'}{X} = -\frac{\dot{Y}}{Y}.$$

In the last equation, the expression on the left involves functions depending only on $x$ while the expression on the right involves functions depending only on $y$. Hence both expressions must be equal to a constant[2], say, $k$.

Hence

$$\frac{X'}{X} = -\frac{\dot{Y}}{Y} = k. \tag{26.8}$$

(26.8) yields two ordinary differential equations, viz,

$$X' - kX = 0. \tag{26.9}$$

and

$$\dot{Y} + kY = 0 \tag{26.10}$$

[2]Reason: If the expression on the left is not constant, then changing $x$ will presumably change the value of this expression but certainly not that on the right, since the latter does not depend on $x$. Similarly, if the expression on the right is not constant, changing $y$ will presumably change the value of this expression but certainly not that on the left.

(26.9) is a linear differential equation in $x$:

$$\frac{dX}{dx} - kX = 0$$

and its solution is

$$X(x) = c_1 e^{kx}.$$

(26.10) is a linear differential equation in $y$:

$$\frac{dY}{dy} + kY = 0$$

and its solution is $Y(y) = c_2 e^{-ky}$.
Hence solution to (26.7) is

$$u(x,\ y) = X(x)Y(y) = Ce^{kx}e^{-ky} = Ce^{k(x-y)}.$$

**Exercises**

In Exercises 1-3, verify that the functions are solutions of the one dimensional wave equation (26.7) for a suitable value of $a$.

1. $u = sin\ \omega t\ sin \omega x$
2. $u = x^2 + \ 4t^2$
3. $u = e^{-t} \sin 3x$

In Exercises 4-6, verify that the functions are solution of the heat equation (26.8) for a suitable value of $\alpha$

4. $u = e^{-2t} \cos x$
5. $u = e^{-t} \sin 3x$
6. $u = e^{-4t} \cos \omega x$

In Exercises 7-12, verify that the following functions are solutions of Laplace's equation (26.9).

7. $u = 2xy$
8. $u = e^{x} \sin\ y$
9. $u = tan\ ^{1}(y/x)$
10. $u = x^3 - 3xy^2$
11. $u = \sin\ x\ \sinh\ y$
12. $u = x^4 - 6x^2y^2 + y^4$
13. State the one dimensional wave equation. Show that the function $u(x,\ t) = v(x + ct) + w(x - ct)$ is a solution of the wave equation, where $u$ and $v$ are any twice differentiable functions.
14. Show that $u = (x^2 + y^2 + z^2)^{-1/2}$ is a solution of the three dimensional Laplace's equation.

In Exercises 15 and 16, solve the following equation where $u$ is a function of two variables $x$ and $y$.

15. $u_x + 2xu = 0$
16. $u_x = 2xyu$

In Exercises 17 and 18, setting $u_x = p$, solve

17. $u_{xy} = 0$
18. $u_{xy} + u_x = 0$

Solve the following systems of partial differential equations.

19. $u_x = 0,\ u_y = 0$

20. . $u_{xx} = 0,\ u_{xy} = 0$

21. . $u_{xx} = 0,\ u_{xy} = 0,\ u_{yy} = 0$

**Answers**

1. $a = \pm 1$

3. $a = \pm\frac{1}{3}i$

5. $a = \pm\frac{1}{3}$

14. Hint: $\frac{\partial^2 u}{\partial x^2} = 3x^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2}$. Similarly, or by symmetry obtain $\frac{\partial^2 u}{\partial y^2},\ \frac{\partial^2 u}{\partial z^2}$.

15. $u = f(y)e^{-x}$

16. $u = f(y)e^{xy}$

17. $u = \int f(x)dx + g(y))$

18. $e^{-y}\int f(x)dx + g(y))$

20. $u = cx + f(y)$

21. $u = ax + by + c$

# Chapter 27

# The Heat Equation

One of the partial differential equations that occur frequently in applied mathematics is the heat equation. We describe the equation and solve the equation under some conditions.

## 27.1 Heat Conduction Equation

We consider the temperature $u$ in a long thin bar or wire of constant cross section and homogeneous material which is oriented along the $x$-axis and is perfectly insulated laterally, so that heat flows in the $x$-direction only. Then $u$ depends only on the axial coordinate $x$ and time $t$, so that $u = u(x,\ t)$. Also suppose that $x = 0$ and $x = L$ are the axial coordinates corresponding to the ends of the bar.

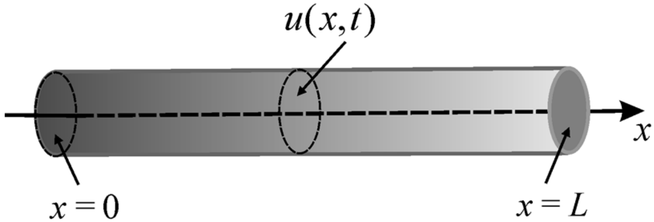


Figure 27.1: A heat-conducting solid bar

The variation of temperature in the bar is governed by the following partial differential equation:

$$\alpha^2 u_{xx} = u_t, \qquad 0 < x < L, \quad t > 0. \tag{27.1}$$

The above equation is called **heat conduction equation** or **one-dimensional heat equation.** In the above $\alpha^2$ depends only on the material from which the bar is made and is defined by

$$\alpha^2 = \frac{\kappa}{\rho s},$$

where $\kappa$ is the thermal conductivity, $\rho$ is the density and $s$ is the specific heat of the material of the body.

**Solution of Heat Conduction Equation with Boundary and Initial Conditions**

Now we shall *solve* the heat equation (27.1) for some important types of *boundary and initial conditions.*

The heat conduction equation is

$$\alpha^2 u_{-xx} = u_t, \qquad 0 < x < L, \quad t > 0. \tag{27.1}$$

Let us start with the case when the ends $x = 0$ and $x = L$ of the bar are kept at temperature zero. Then the *boundary conditions* are

$$u(0,\ t) = 0, \qquad u(L,\ t) = 0, \qquad t > 0. \tag{27.2}$$

Let $f(x)$ be the initial temperature (i.e., temperature at time$t = 0$) in the bar at the point with axial coordinate $x$. Then the *initial condition* is

$$u(x,\ 0) = f(x). \tag{27.3}$$

*We shall determine a solution* $u(x,\ t)$ *of* (27.1) *satisfying* (27.2) *and* (27.3).

We assume that $u(x,t) = X(x)\,T(t)$.

Then using (27.2), for $t > 0$

$$u(0,\ t) = X(0)T(t) = 0 \qquad u(L,\ t) = X(L)T(t) = 0.$$

i.e.,

$$X(0)T(t) = 0\ = X(L)T(t) \ \text{ for } \ t > 0.$$

The above says that **either** $T(t) = 0$ for $t > 0$ **or** $X(0) = 0 = X(L)$. The first case is of no interest as $T(t) = 0$ implies $u(x,\ t) = X(x)T(t) = X(x)0 = 0$ which means the temperature on the bar

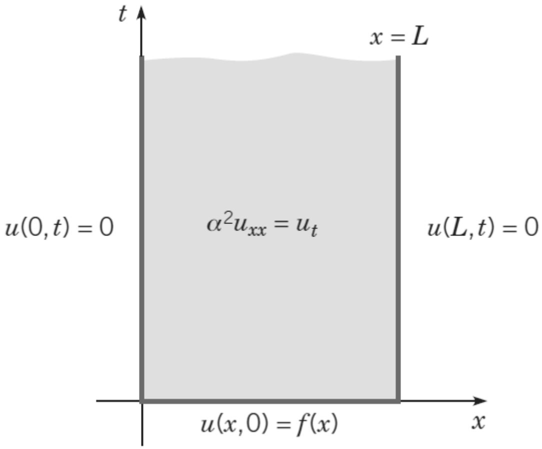


Figure 27.2: Boundary value problem for the wave equation.

is always zero. Hence hereinafter we consider the case

$$X(0) = 0 = X(L). \tag{27.4}$$

Now we shall proceed step by step, as follows.

**Step 1.** By applying the product method[1], or *method of separating variables*, we shall obtain two ordinary differential equation.

**Step 2.** We shall determine solutions of those two equations that satisfy the boundary conditions.

**Step 3.** Those solutions will be composed so that the result will be a solution of the heat equation (27.1), satisfying also the given initial condition.

[1]Product Method (or Method of Separating Variables) is discussed in the previous chapter.

The details are as follows.

**Step 1** Applying the method of separation of variables we first determine solutions of (27.1) that satisfy the boundary conditions (27.2). We start from

$$u(x,\ t) = X(x)T(t). \tag{27.5}$$

Substituting this expression into (27.1), we obtain the equation

$$\alpha^2 X'' T = X\dot{T}$$

where primes denote derivatives with respect to $x$ and dots denote derivatives with respect to $t$. To separate variables, we divide the equation by $\alpha^2 X\, T$, and obtain

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{\dot{T}}{T}.$$

The expression on the left depends only on $t$, while the right side depends only on $x$. We conclude that both expressions must be equal to a constant, say, $k$. [2]

That is,

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{\dot{T}}{T} = k.$$

[2]REASON: If the expression on the left is not constant, then changing $x$ will presumably change the value of this expression but certainly not that on the right, since the latter does not depend on $x$. Similarly, if the expression on the right is not constant, changing $y$ will presumably change the value of this expression but certainly not that on the left.

Hence we obtain two ordinary differential equations,

$$X'' - kX = 0 \tag{27.6}$$

and

$$\dot{T} - \alpha^2 kT = 0. \tag{27.7}$$

Case 1) If $k = 0$, then (27.6) becomes

$$X'' = 0.$$

Integrating (with respect to $x$),

$$X' = a,$$

where $a$ is an arbitrary constant. One more integration (with respect to $x$) yields

$$X = X(x) = ax + b,$$

where $b$ is also an arbitrary constant.

By (27.4), $X(0) = 0$, and hence the above implies $b = 0$. Hence the above reduces to

$$X = X(x) = ax.$$

Again by (27.4), $X(L) = 0$, hence the above implies $aL = 0$ which implies $a = 0$. Hence the above reduces to

$$X = X(x) = 0.$$

This means that

$$u(x,\ t) = X(x)T(t) = 0T(t) = 0$$

which means the temperature on the bar is always zero. Hence we ruled out the case $k = 0$.

Case 2) If $k > 0$, say $k = \mu^2$, then the characteristic equation corresponding to the second order ordinary differential equation (27.6) is

$$\lambda^2 - \mu^2 = 0$$

i.e.,

$$\lambda^2 = \mu^2.$$

Hence the general solution of (27.6) is

$$X(x) = Ae^{\mu x} + Be^{-\mu x}.$$

For the convenience of applying boundary conditions, recalling the definitions of hyperbolic sine and cosine functions, we write the above general solution as

$$X(x) = c_1 \cosh \mu\, x + c_2 \sinh \mu\, x.$$

By (27.4), $X(0) = 0$, and hence the above implies

$$c_1 = 0.$$

Thus

$$X(x) = c_2 \sinh \mu x.$$

Also by (2A), $X(L) = 0$, and hence the above implies

$$c_2 \sin \mathrm{h}\, \mu L = 0$$

Since $\mu \neq 0,\ \sinh\ \mu L \neq 0,$, hence we have $c_2 = 0.$ Hence

$$X(x) = 0.$$

This means that

$$u(x,\ t) = X(x)T(t) = 0T(t) = 0$$

which means the temperature on the bar is always zero. Hence we ruled out the case $k > 0$.

Case 3) Hence *we are left with the possibility* of choosing $k$, negative, say, $k = -\lambda$, with $\lambda >$ 0then equations (27.6) and (27.7) takes the form

$$X'' + \lambda X = 0 \tag{27.8}$$

and

$$\dot{T} + \alpha^2 \lambda T = 0. \tag{27.9}$$

If we take $\lambda = \mu^2$, then $k = -\mu^2$, so that we can also write equations (27.6) and (27.7) in the form

$$X'' + \mu^2 X = 0 \tag{27.10}$$

and

$$\dot{T} + \alpha^2\mu^2 T = 0. \tag{27.11}$$

***Step* 2** Consider equation (27.10). Its general solution is

$$X(x) = A\cos\mu x + B\sin\mu x. \tag{27.12}$$

From the boundary conditions (27.4), it follows that

$$X(0) = A = 0\,.$$

Hence

$$X(x) = B\sin\mu x.$$

Again by (27.4),

$$X(L) = B\sin\mu L = 0.$$

Two cases arise: either $B = 0$ or $\sin\mu L = 0$.

The case of $B = 0$ leads to $X(x) = 0$ for $0 \le x \le L$, so that

$$u(x,\ t) = X(x)T(t) = X(x)0 = 0$$

which is no of interest. Hence we take

$$\sin \mu L = 0.$$

This implies

$$\mu L = n\pi \,, \quad n = 1,\ 2,\ 3,\ \ldots$$

or

$$\mu = \frac{n\pi}{L}, \quad n = 1,\ 2,\ 3,\ \ldots$$

Hence the only nontrivial solutions of Eq. (27.10) with boundary conditions (2A) and (2B) are the eigen functions.[3]

$$X_n(x) = \sin \frac{n\pi x}{L} \quad , \quad n = 1,\ 2,\ 3,\ \ldots$$

associated with the eigen values

$$\lambda_n = \frac{n^2\pi^2}{L^2} \quad , \quad n = 1,\ 2,\ 3,\ \ldots \tag{27.13}$$

Substituting $\frac{n^2\pi^2}{L^2}$ for $\lambda$, Eq.(27.13) takes the form

$$\dot{T} + \frac{n^2\pi^2\alpha^2}{L^2} T = 0. \tag{27.14}$$

The general solution of the first order differential equation (27.14)

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[3] Ref. section eigen value problems in chapter “ Two point Boundary Value Problems”.

is

$$T_n(t) = B_n \exp\left(-\frac{n^2\pi^2\alpha^2 t}{L^2}\right), \quad n = 1,\ 2,\ \ldots \tag{27.15}$$

where $B_n$ is arbitrary constant. Neglecting the arbitrary constants of proportionality, we conclude that the functions

$$u_n(x,\ t) = e^{-n^2\pi^2\alpha^2 t/L^2} \sin\frac{n\pi x}{L}, \quad n = 1,\ 2,\ \ldots \tag{27.16}$$

are solutions of the heat equation (27.1), satisfying the boundary conditions (27.2).

***Step* 3** Now we have to obtain a solution that also satisfying the initial condition (27.3)

$$u(x,\ 0) = f(x).$$

For this we form linear combinations of a set of fundamental solutions and then choose the coefficients to satisfy the initial conditions. Hence we form a linear combination of the functions (27.16) and then choose the coefficients to satisfy Eq.(27.3). Here there are infinitely many functions (27.16), so a general linear combination of them is an infinite series of the following form

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,\ t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2\alpha^2 t/L^2} \sin\frac{n\pi x}{L} \ldots \tag{27.17}$$

where the coefficients are to be determined.

We assume that the infinite series of Eq.(27.17) converges and also satisfies Eqs.(27.1) and (27.2). To satisfy the initial condition (27.3), we must have (with $t = 0$).

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L} = f(x)\ldots \tag{27.18}$$

Hence, for (27.18) to satisfy (27.3), the coefficients $c_n$ must be chosen such that $u(x,0)$ becomes a half-range expansion of$f(x)$, namely, the Fourier sine series of$f(x)$. In that case,

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx, \qquad n = 1,\ 2,\ \ldots\ldots \tag{27.19}$$

The solution of our problem can be established, assuming that $f(x)$ is piecewise continuous on the interval$0 \leq x \leq L$, and has one-sided derivatives at all interior points of that interval. Under these assumptions, the series (27.17) with coefficients given by (27.19) is the **solution** of the given heat problem with the given boundary and initial conditions.

***Example* 1** Find the temperature $u(x,t)$ in a long thin bar of length $L$ units with uniform cross section and homogeneous material and is insulated laterally, so that heat flows only in the direction of the bar, if the ends $x = 0$ and $x = L$ of the bar are kept at zero temperature and the initial temperature is

$$f(x) = \begin{cases} x & \text{when} \quad 0<x<\frac{L}{2} \\ L-x & \text{when} \quad \frac{L}{2}<x<L \end{cases}$$

*Solution*

$$c_n = \frac{2}{L}\left\{\int_0^{L/2} x\sin\frac{n\pi x}{L}dx + \int_{L/2}^{L}(L-x)\sin\frac{n\pi x}{L}dx\right\}$$

$= \frac{4L}{n^2\pi^2}\sin\frac{n\pi}{2}$, on simplification.

Hence

$$c_n = \begin{cases} \frac{4L}{n^2\pi^2}, & \text{when} \quad n = 1, 5, 9, ... \\ -\frac{4L}{n^2\pi^2}, & \text{when} \quad n = 3, 7, 11, .. \\ 0, & \text{when} \quad n \text{ is even} \end{cases}.$$

Putting the above values of $c_n$ in (27.17), the solution is

$$u(x,t) = \frac{4L}{\pi^2}\left[\sin\frac{\pi x}{L}e^{-\pi^2\alpha^2 t/L^2} - \frac{1}{9}\sin\frac{3\pi x}{l}e^{-9\pi^2\alpha^2 t/L^2} + \quad . \quad . \quad . \quad\right]$$

***Example* 2** Find the temperature $u(x,t)$ in a metal rod of length 50 cm long with uniform cross section and homogeneous material and is insulated laterally, so that heat flows only in the direction of the bar, if the ends of the bar are kept at zero temperature and the initial temperature is 20°C.

*Solution*

Here $L = 50$ and $f(x) = 20$ for $0 < x < 50$. Thus,

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2\alpha^2 t/2500}\sin\frac{n\pi x}{50}$$

where

$$c_n = \frac{2}{50}\int_0^{50} 20\sin\frac{n\pi x}{50}dx$$

$$= \frac{40}{n\pi}(1 - \cos n\pi)$$

$$= \begin{cases} \frac{80}{n\pi}, \text{ when } n \text{ odd} \\ 0, \quad \text{when} \quad n \text{ even} \end{cases}$$

Hence

$$u(x,\ t) = \frac{80}{\pi}\left[\sin\frac{\pi x}{50}e^{-\pi^2\alpha^2 t/2500} + \frac{1}{3}\sin\frac{3\pi x}{50}e^{-9\pi^2\alpha^2 t/2500} + \cdots\right]$$

**Exercises**

In Exercises 1-4, find the temperature $u(x,t)$ in a bar of silver (length 10 cm, constant cross section of area 1 cm$^2$, density 10.6 gm/cm$^3$, thermal conductivity 1.04. cal/cm deg sec. Specific heat 0.056 cal/gm deg) which is perfectly insulated laterally, whose ends are kept at temperature $0^o$C, and whose initial temperature (in $^o$C) is $f(x)$, where

1. $f(x) = \sin 0.1x$

2. $f(x) = \begin{cases} x, \ if \quad 0 < x < 5 \\ 10 - x, \ if \quad 5 < x < 10 \end{cases}$

3. $f(x) = \begin{cases} x, \ if \quad 0 < x < 5 \\ 0, \ if \quad 5 < x < 10 \end{cases}$

4. $f(x) = x(10 - x)$

In Exercises 5-9, find the temperature $u(x,\ t)$ in a bar of length $L = \pi$ which is perfectly insulated, also at the ends at $x = 0$ and $x = L$ assuming that $u(x,0) = f(x)$, where $f(x)$ is given by

5. $f(x) = 1$

6. $f(x) = x^2$

7. $f(x) = 0.5\cos 2x$

8. $f(x) = \sin x$

9. $f(x) = \begin{cases} x, & \text{if } 0 < x < \pi/2 \\ \pi - x, & \text{if } \pi/2 < x < \pi. \end{cases}$

Consider the conduction of heat in a rod 40 cm in length whose ends are maintained at 0°C for all $t > 0$. In each of Exercises 10 through 13 find an expression for the temperature $u(x,\ t)$ if the initial temperature distribution in the rod is the given function. Suppose that $\alpha^2 = 1$.

10. $u(x,\ 0) = 50, \qquad 0 < x < 40$

11. $u(x,\ 0) = \begin{cases} x, & 0 \le x < 20 \\ 40 - x, & 20 \le x \le 40 \end{cases}$

12. $u(x,\ 0) = x, \qquad 0 < x < 40$

13. $u(x,\ 0) = \begin{cases} 0, & 0 \le x < 10 \\ 50, & 10 \le x \le 30 \\ 0, & 30 < x \le 40 \end{cases}$

14. Find the solution of the heat conduction problem

$$u_{xx} = 4u_t, \qquad 0 < x < 2, \quad t > 0;$$

$$u(0,\ t) = 0, \qquad u(2,\ t) = 0, \quad t > 0;$$

$$u(x,\ 0) = 2\sin(\pi x/2) - \sin \pi x + 4\sin 2\pi x, \qquad 0 \le x \le 2.$$

15. Find the solution of the heat conduction problem

$$100u_{xx} = u_t, \qquad 0 < x < 1, \quad t > 0;$$

$$u(0,\ t) = 0, \qquad u(1,\ t) = 0, \quad t > 0;$$

$$u(x,\ 0) = 2(\sin 2\pi x - \sin 5\pi x), \qquad 0 \le x \le 1.$$

***Answers***

1. $u(x,t) = \sin 0.1\pi x e^{-1.752\pi^2 t/100}$
2. $u(x,t) = \frac{40}{\pi^2}\left(\sin 0.1\pi x e^{-0.0175\pi^2 t} - \frac{1}{9}\sin 0.3\pi x e^{-0.0175(3\pi)^2 t} + ...\right)$
4. $u(x,t) = \frac{800}{\pi^3}\left(\sin 0.1\pi x e^{-0.0175\pi^2 t} + \frac{1}{3^3}\sin 0.3\pi x e^{-0.0175(3\pi)^2 t} + - ...\right)$
5. $u(x,\ t) = 1$
7. $0.5\cos 2x\ e^{-4t}$
9. $u(x,t) = \frac{\pi}{4} - \frac{8}{\pi}\left(\frac{1}{4}\cos 2te^{-4t} + \frac{1}{36}\cos 6te^{-36t} + ...\right)$
10. $u(x,t) = \frac{100}{\pi}\sum_{n=1}^{\infty}\frac{1-\cos n\pi}{n}e^{-n^2\pi^2 t/1600}\sin\frac{n\pi x}{40}$

11. $u(x,t) = \frac{160}{\pi^2}\sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n^2} e^{-n^2\pi^2 t/1600} \sin\frac{n\pi x}{40}$

12. $u(x,t) = \frac{80}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} e^{-n^2\pi^2 t/1600} \sin\frac{n\pi x}{40}$

13. $u(x,t) = \frac{100}{\pi}\sum_{n=1}^{\infty} \frac{\cos(n\pi/4)-\cos(3n\pi/4)}{n} e^{-n^2\pi^2 t/1600} \sin\frac{n\pi x}{40}$

14. $u(x,t) = 2e^{-\pi^2 t/16} \sin(\pi x/2) - e^{-\pi^2 t/4} \sin\pi x + 4e^{-\pi^2 t} \sin 2\pi x$

15. $u(x,t) = (e^{-400\pi^2 t} \sin 2\pi x - e^{-2500\pi^2 t} \sin 5\pi x) \cdot 2$

Chapter 28

# Vibrating String-Wave Equation

## 28.1 Vibrating String-Wave Equation

One of the partial differential equations that occurs frequently in applied mathematics is the wave equation. In this chapter we first describe one dimensional wave equation and then consider its solution.

Suppose that an elastic string of the length $L$ is tightly stretched between two supports at the same horizontal level, so that the $x$ -axis lies along the string (Fig. 28.1) Let $u(x,\ t)$ denote the vertical displacement experienced by the string at the point $x$ at time $t$. We also make the following assumptions.

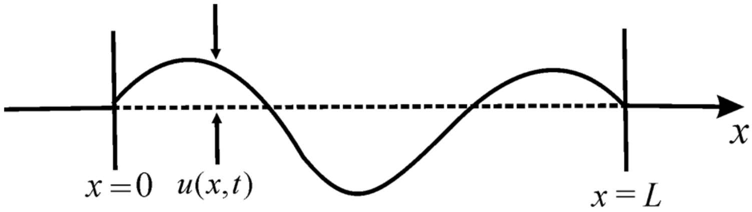


Figure 28.1: A vibrating string

1. The mass of the string per unit length is constant (homogeneous string). The string is perfectly elastic and does not offer any resistance to bending.

2. The tension caused by stretching the string before fixing it at the endpoints is so large that the action of the gravitational force on the string can be neglected.

3. The string performs a small transverse motion in a vertical plane, that is, every particle of the string moves strictly vertically and so that the deflection and the slope at every point of the string remain small in absolute value.

Then $u(x,\ t)$ satisfies the partial differential equation

$$a^2 u_{xx} = u_{tt} \tag{28.1}$$

in the domain $0 < x < L$, $t > 0$. Eq.(28.1) is known as the **one-dimensional wave equation.** The constant coefficient $a^2$

in Eq.(28.1) is given by

$$a^2 = \frac{T}{\rho},$$

where $T$ is the tension (force) in the string, and $\rho$ is the mass per unit length of the string material. $a$ has the units of length/time; i.e., $a$ has the units of velocity.

Since the string is fixed at the ends $x = 0$ and $x = L$, we have the two **boundary conditions**

$$u(0,\ t) = 0, \quad u(L,\ t) = 0 \ \text{ for } t > 0. \tag{28.2}$$

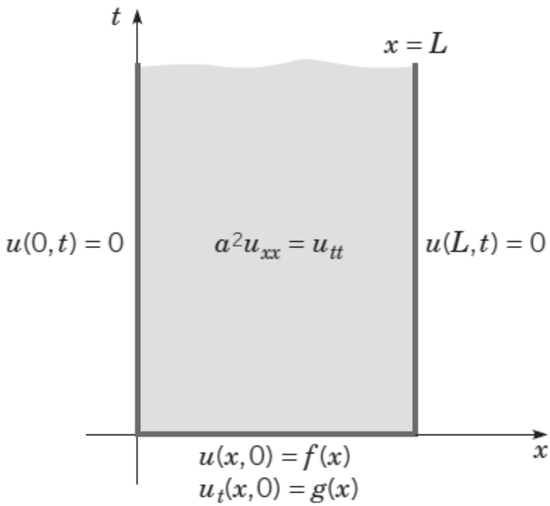


Figure 28.2: Boundary value problem for the wave equation.

The form of the motion of the string will depend on the initial deflection (deflection at time $t = 0$) and on the initial velocity (velocity at time $t = 0$.) Denoting the **initial deflection** by $f(x)$ and the **initial velocity** by $g(x)$, we have the two **initial conditions**

$$u(x,\ 0) = f(x), \quad 0 \le x \le L \tag{28.3}$$

and

$$u_t(x,\ 0) = g(x), \quad 0 \le x \le L. \tag{28.4}$$

In order for Eqs. (28.2), (28.3), and (28.4) to be consistent, it is also necessary to require that

$$f(0) = f(L) = 0, \qquad g(0) = g(L) = 0. \tag{28.5}$$

**Solution to the wave equation by separation of variables (Product Method)**

We now find the value of $u(x,\ t)$ that satisfy the *one-dimensional wave equation* given by

$$a^2 u_{xx} = u_{tt} \qquad (28.1)$$

and satisfying the two boundary conditions given by

$$u(0,\ t) = 0, \qquad u(L,\ t) = 0 \text{ for } t > 0. \tag{28.6}$$

and the two initial conditions given by

$$u(x,\ 0) = f(x),\ 0 \le x \le L \tag{28.7}$$

and

$$u_t(x,\ 0) = 0,\ 0 \leq x \leq L. \tag{28.8}$$

[Eq.(28.8) means the initial velocity is 0. Here $g(x) = 0$.]

**Step 1**. The product method yields solutions of equation (28.1) of the form

$$u(x,\ t) = X(x)T(t) \tag{28.9}$$

which is a product of two functions $X(x)$and $T(t)$, each depending only one of the variables $x$ and $t$, respectively.

Differentiating (28.9) twice with respect to$x$, we obtain

$$u_{xx} = X''T,$$

where primes denote derivatives with respect to $x$. Differentiating (28.9) twice with respect to$t$, we obtain

$$u_{tt} = X\ddot{T},$$

where dots denote derivatives with respect to $t$. By inserting this into the differential equation (28.1), we obtain

$$a^2X''T = X\ddot{T}$$

Division by $a^2XT$ yields

$$\frac{X''}{X} = \frac{1}{a^2}\frac{\ddot{T}}{T}.$$

The expression on the left involves function depending only on $t$ while the expression on the right involves function depending only on $x$. Hence both expression must be equal to a constant. Thus,

$$\frac{X''}{X} = \frac{1}{a^2}\frac{\ddot{T}}{T} = -\lambda,$$

where $\lambda$ is arbitrary. This yield immediately two second order ordinary linear differential equations, viz.

$$X'' + \lambda X = 0 \tag{28.10}$$

and

$$\ddot{T} + a^2\lambda T = 0. \tag{28.11}$$

**Step 2** We shall now determine solutions $X$ and $T$ of (28.10) and (28.11) so that $u$ given by (28.9) *satisfies the boundary conditions* (28.6).

By substituting from Eq.(28.9) for $u(x,\ t)$ in the boundary conditions (28.6) (28.7), we find that $X(x)$must satisfy the boundary conditions

$$X(0) = 0, \qquad X(L) = 0. \tag{28.12}$$

Also, by substituting from Eq.(28.9) for $u(x,\ t)$ in the initial condition (28.8), we find that $T(t)$must satisfy the initial condition

$$X(0) = 0, \qquad X(L) = 0. \tag{28.13}$$

**Step3**. We now determine $X(x)$, $T(t)$, and $\lambda$ by solving Eq.

(28.10) subject to the boundary conditions (28.12) and Eq.(28.11) subject to the initial condition (28.13).

The problem of solving the differential equation (28.10) subject to the boundary conditions (28.12) is precisely the same problem that arose in the previous chapter in connection with heat conduction equation. Thus we can use the results obtained there and at the end of the chapter *Two Point Boundary Value Problems*. Using those results, the problem (28.10) subject to (28.12) has **nontrivial solutions if and only if $\lambda$ is an eigen value**

$$\lambda = \frac{n^2\pi^2}{L^2} \text{ for } n = 1,\ 2,\ 3,\ \ldots \tag{28.14}$$

and $X(x)$is proportional to the corresponding eigen function $\sin\frac{n\pi x}{L}$.

Using the values of $\lambda$ given by Eq.(28.14) in Eq.(28.11), we obtain

$$\ddot{T} + \frac{n^2\pi^2 a^2}{L^2} T = 0. \tag{28.15}$$

Hence

$$T(t) = k_1 \cos\frac{n\pi at}{L} + k_2 \sin\frac{n\pi at}{L}, \tag{28.16}$$

where $k_1$ and $k_2$ are arbitrary constants. The initial condition (28.13) requires that $k_2 = 0$. Hence

$$T(t) = k_1 \cos\frac{n\pi at}{L}.$$

Neglecting the arbitrary constants of proportionality, we conclude that the functions

$$u_n(x,\ t) = \sin\frac{n\pi x}{L}\cos\frac{n\pi at}{L}, \quad n = 1,\ 2,\ 3,\ \ldots \tag{28.17}$$

satisfy the partial differential equation (28.1), the boundary conditions (28.6), and the initial condition (28.8). These functions are the *fundamental solutions* for the given problem.

To satisfy the remaining (nonhomogeneous) initial condition (28.7), we consider a superposition (linear combination) of the fundamental solutions (28.8) with properly chosen coefficients. Thus we assume that
$u(x,\ t)$ has the form

$$u(x,\ t) = \sum_{n=1}^{\infty} c_n u_n(x,\ t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}\cos\frac{n\pi at}{L}, \tag{28.18}$$

where the constants $c_n$ are to be determined. The initial condition (28.7) requires that

$$u(x,\ 0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L} = f(x). \tag{28.19}$$

Hence, in order that (28.18) satisfy (28.7), the coefficients $c_n$ must be chosen so that $u(x,\ 0)$becomes a **half-range expansion** of$f(x)$, namely, the **half range Fourier sine series** of$f(x)$; we obtain the Fourier coefficients as

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1,\ 2,\ 3,\ \ldots \tag{28.20}$$

It follows that $u(x,\ t)$ given by (28.18), with coefficients (28.20) is a **solution** of (28.1) that satisfies the conditions (28.6) to (28.8), provided that the series (28.18) converges and also that the series obtained by differentiating (28.18) twice (term wise) with respect to $x$ and $t$ converge and have the sums $u_{xx}$ and $u_{tt}$ respectively, which are continuous.

For a fixed value of $n$ the expression $\cos\frac{n\pi at}{L}$ in Eq.(28.17) is periodic in time with the period $\frac{2L}{na}$. The quantities $\frac{n\pi a}{L}$ for $n = 1,\ 2,\ 3,\ \ldots$ are the **natural frequencies** of the string – that is, the frequencies at which the string will freely vibrate. The factor $\sin\frac{n\pi x}{L}$ represents the displacement pattern occuring in the string when it is executing vibrations of the given frequency. Each displacement pattern is called a **natural mode** of vibration and is periodic in the space variable $x$; the spatial period $\frac{2L}{n}$ is called the **wavelength** of the mode of frequency $\frac{n\pi a}{L}$. Thus the eigen values $\frac{n^2\pi^2}{L^2}$ are proportional to the squares of the natural frequencies, and the eigen functions $\sin\frac{n\pi x}{L}$ give the natural modes.

***Example* 1** (*Vibrating string if the initial deflection is triangular*)

Find the solution of the wave equation (28.1) subject to the conditions (28.6) to (28.8), where it is given the triangular initial deflection

$$f(x) = \begin{cases} \frac{2k}{L}x, & \text{when} \quad 0<x<\frac{L}{2} \\ \frac{2k}{L}(L-x), & \text{when} \quad \frac{L}{2}<x<L \end{cases}$$

and the initial velocity zero.

We solve the problem by finding $u(x,\ t)$ given by (28.18), with coefficients (28.20):

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

$$= \frac{2}{L}\left\{\int_0^{L/2}\frac{2k}{L}x\sin\frac{n\pi x}{L}dx + \int_{L/2}^{L}\frac{2k}{L}(L-x)\sin\frac{n\pi x}{L}dx\right\}$$

$= \frac{8k}{\pi^2 n^2}\sin\frac{n\pi}{2}$, on simplification.

Hence,

$$u(x,t) = \sum_{n=1}^{\infty}\frac{8k}{\pi^2 n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{L}\cos\frac{n\pi at}{l}$$

i.e.,

$$u(x,t) = \frac{8k}{\pi^2}\left[\frac{1}{1^2}\sin\frac{\pi x}{L}\cos\frac{\pi at}{L} - \frac{1}{3^2}\sin\frac{3\pi x}{L}\cos\frac{3\pi at}{L} + \cdots\right]$$

***Example* 2** A string is stretched and fastened at two points $L$ apart. Motion is started by displacing the string in the form

$$u = u(x,t) = a\sin\frac{\pi x}{l}$$

from which it is released at time $t = 0$. Show that the displacement at any point at a distance $x$ from one end at time $t$ is given by

$$u = a\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l}.$$

*Solution*

Here initial deflection is

$$u(x,0)=f(x)=a\sin\frac{\pi x}{l},$$

and initial velocity is

$$g(x)=0.$$

Hence, we solve the problem by finding $u(x,\ t)$ given by (28.18), with coefficients (28.20):

$$c_n=\frac{2}{L}\int_0^L a\sin\frac{\pi x}{l}\sin\frac{n\pi x}{l}dx$$

$$=\begin{cases} a & \text{for} \quad n=1 \\ 0 & \text{for} \quad n\neq 1 \end{cases}, \text{ on simplification.}$$

Substituting these values in (28.18), we obtain

$$u=u(x,\ t)=a\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l},$$

proving the result.

***Example* 3** Consider a vibrating string of length $L=30$ that satisfies the wave equation

$$4u_{xx}=u_{tt}, \qquad 0<x<30, \quad t>0.$$

Assume that the ends of the string are fixed and that the string is set in motion with no initial velocity from the initial position

$$u(x,\ 0) = f(x) = \begin{cases} \frac{x}{10}, & 0 \le x \le 10 \\ \frac{30-x}{20}, & 10 < x \le 30 \end{cases}$$

Find the displacement $u(x,\ t)$of the string and describe its motion through one period.

*Solution*

Here $a = 2$ and $L = 30$. Hence the solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{30} \cos\frac{2n\pi \mathrm{t}}{30}$$

where

$$c_n = \frac{2}{30}\left\{\int_0^{10} \frac{x}{10}\sin\frac{n\pi x}{30}dx + \int_{10}^{30} \frac{30-x}{20}\sin\frac{n\pi x}{30}dx\right\}$$

$= \frac{9}{n^2\pi^2}\sin\frac{n\pi}{3}, \quad n = 1,\ 2,\ \ldots$ (on simplification)

**Justification of the Solution**

The solution given by (28.18) with coefficients $c_n$ given by (28.20) is purely a formal expression. Now for the justification of whether Eq.(28.18) actually represents the solution of the given problem requires some further investigation. We establish the validity of Eq.(28.18) indirectly.

First we will show that Eq.(28.18) is equivalent to

$$u(x,t) = \frac{1}{2}\left[h(x-at) + h(x+at)\right], \tag{28.21}$$

where $h$ is the odd $2L$ periodic extension of the function $f$. [This is obtained as follows: First take the odd extension to $(-L,\ L)$ of the function $f$ defined on $[0,\ L]$. Then extend the so obtained function to the entire real line by defining at other values of $x$ as a period function of period $2L$.] That is,

$$h(x) = \begin{cases} f(x), & 0 \leq x \leq L, \\ -f(-x), & -L < x < 0; \end{cases} \tag{28.22}$$

and

$$h(x + 2L) = h(x).$$

Being the odd $2L$ periodic extension of the function $f$, the function $h$ has the half range Fourier sine series representation of $f(x)$. Hence, using Eq.(28.19), we can write $h$ as

$$h(x) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L}, \tag{28.23}$$

where $c_n$ is given by (28.20)

$$c_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx, \quad n = 1,\ 2,\ 3,\ \ldots$$

Now recalling the trigonometric identity

$$\sin(A - B) = \sin A \cos B - \cos A \sin B,$$

we have

$$h(x-at)=\sum_{n=1}^{\infty}c_n\sin\frac{n\pi(x-at)}{L}=\sum_{n=1}^{\infty}c_n\sin\left(\frac{n\pi x}{L}-\frac{n\pi at}{L}\right)$$

$$=\sum_{n=1}^{\infty}c_n\left(\sin\frac{n\pi x}{L}\cos\frac{n\pi at}{L}-\cos\frac{n\pi x}{L}\sin\frac{n\pi at}{L}\right)$$

and using the trigonometric identity

$$\sin(A+B)=\sin A\cos B+\cos A\sin B,$$

we have

$$h(x+at)=\sum_{n=1}^{\infty}c_n\sin\frac{n\pi(x+at)}{L}=\sum_{n=1}^{\infty}c_n\sin\left(\frac{n\pi x}{L}+\frac{n\pi at}{L}\right)$$

$$=\sum_{n=1}^{\infty}c_n\left(\sin\frac{n\pi x}{L}\cos\frac{n\pi at}{L}+\cos\frac{n\pi x}{L}\sin\frac{n\pi at}{L}\right).$$

Adding the above two equations, we obtain

$$h(x-at)+h(x+at)=2\sum_{n=1}^{\infty}c_n\sin\frac{n\pi x}{L}\cos\frac{n\pi at}{L},$$

so using (28.18), we have

$$h(x-at)+h(x+at)=2u(x,t),$$

and this establishes Eq.(28.21).

From Eq.(28.21) we see that $u(x,t)$ is continuous for $0 < x < L,\ t > 0$, provided that $h$ is continuous on the interval $(-\infty,\ \infty)$. This requires $f$ to be continuous on the original interval $[0,\ L]$. Similarly, $u$ is twice continuously differentiable with respect to either variable in $0 < x < L,\ t > 0$, provided that $h$ is twice continuously differentiable on $(-\infty,\ \infty)$. This requires $f'$ and $f''$ to be continuous on $[0,\ L]$. Furthermore, since $h''$ is the odd extension of $f''$, we must also have $f''(0) = f''(L) = 0$. However, since $h'$ is the even extension of $f'$, no further conditions are required on $f'$. Provided that these conditions are met, $u_{xx}$ and $u_{tt}$can be computed from Eq.(28.21), and it can be shown that these derivatives satisfy the wave equation.

**Solution to the General Problem for the Elastic String**

1. We now find the value of $u(x,\ t)$ that satisfy the *one-dimensional wave equation* given by

$$a^2 u_{xx} = u_{tt}$$

satisfying the two boundary conditions given by

$u(0,\ t) = 0, \qquad u(L,\ t) = 0$ for $t > 0$.

and the two initial conditions given by

$$u(x,\ 0) = 0, \quad 0 \leq x \leq L \tag{28.24}$$

and

$$u_t(x,\ 0) = g(x), \quad 0 \leq x \leq L. \tag{28.25}$$

where $g(x)$is the initial velocity at the point $x$ of the string.

The solution of the new problem can be obtained by following the procedure described above for the problem (28.1), satisfying (28.6), (28.7), and (28.8). By separating variables, we find that the problem for $X(x)$is exactly the same as before. Thus, once again, $\lambda = \frac{n^2\pi^2}{L^2}$ and $X(x)$ is proportional to the corresponding eigen function $\sin \frac{n\pi x}{L}$. The differential equation for $T(t)$is again Eq.(28.15)

$$\ddot{T} + \frac{n^2\pi^2 a^2}{L^2} T = 0,$$

but with the initial condition

$$T(0) = 0, \;\; 0 \leq x \leq L. \tag{28.26}$$

corresponding to the initial condition $u(x,\; 0) = 0$.

The general solution of Eq.(28.15) is given by (28.16)

$$T(t) = k_1 \cos \frac{n\pi a t}{L} + k_2 \sin \frac{n\pi a t}{L},$$

but now the initial condition (28.26) requires that $k_1 = 0$. Hence

$$T(t) = k_2 \sin \frac{n\pi a t}{L},$$

and hence the fundamental solutions for the problem (28.1), (28.6), and (28.7) are

$$u_n(x,\; t) = \sin \frac{n\pi x}{L} \sin \frac{n\pi a t}{L}, \quad n = 1,\; 2,\; 3,\; \ldots \tag{28.27}$$

Each of the functions $u_n(x,\ t)$ satisfies the wave equation (28.1), the boundary conditions (28.7), and the initial condition (28.24).

To satisfy the remaining initial condition (28.25), we assume that $u(x,\ t)$can be expressed as a linear combination of the fundamental solutions (28.27); that is,

$$u(x,\ t) = \sum_{n=1}^{\infty} k_n u_n(x,\ t) = \sum_{n=1}^{\infty} k_n \sin\frac{n\pi x}{L} \sin\frac{n\pi at}{L}. \qquad (28.28)$$

To determine the values of the coefficients $k_n$, we differentiate Eq. (28.28) with respect to $t$, set $t = 0$, and use the initial condition (28.25), and obtain the equation

$$u_t(x,\ 0) = \sum_{n=1}^{\infty} \frac{n\pi a}{L} k_n \sin\frac{n\pi x}{L} = g(x). \qquad (28.29)$$

The above is Fourier sine series of period $2L$ for the function$g$. Hence the coefficients are given by

$$\frac{n\pi a}{L} k_n = \frac{2}{L}\int_0^L g(x) \sin\frac{n\pi x}{L} dx, \quad n = 1,\ 2,\ 3,\ \ldots \qquad (28.30)$$

Thus Eq.(28.28) , with the coefficients given by Eq.(28.30), constitutes a solution to the problem of Eqs.(28.1), (28.6), (28.24) and (28.25).

**Use of Principle of Superposition**

We now find the value of $u(x,\ t)$ that satisfy the *one-dimensional* *wave equation* given by

$$a^2 u_{xx} = u_{tt}$$

satisfying the two boundary conditions given by

$u(0,\ t) = 0, \quad u(L,\ t) = 0$ for $t > 0.$

and the two initial conditions given by

$$u(x,\ 0) = f(x), \quad 0 \leq x \leq L \tag{28.31}$$

and

$$u_t(x,\ 0) = g(x), \quad 0 \leq x \leq L. \tag{28.32}$$

where $f(x)$ and $g(x)$ are the initial position and velocity, respectively, at the point $x$ of the string.

Although this problem can be solved by separating variables, as in the cases discussed so far, it is important to note that it can also be solved simply by adding together the two solutions that we obtained above (This is possible by the use of principle of superposition.) To verify that this is true, let $v(x,\ t)$ be the solution of the problem (28.1), satisfying (28.6), (28.7), and (28.8), and let $w(x,\ t)$ be the solution of the problem of Eqs.(28.1), (28.6), (28.24) and (28.25). Thus $v(x,\ t)$ is given by (28.18), with coefficients (28.20), and $w(x,\ t)$ is given by Eq.(28.27), with the coefficients given by Eq.(28.30).

Now let

$$u(x,\ t) = v(x,\ t) + w(x,\ t).$$

First, we observe that

$$a^2u_{xx} - u_{tt} = (a^2v_{xx} - v_{tt}) + (a^2w_{xx} - w_{tt}) = 0 + 0 = 0, \quad (28.33)$$

so $u(x,\ t)$satisfies the wave equation (28.1). Next, we have

$$\left.\begin{aligned} &u(0,\ t) = v(0,\ t) + w(0,\ t) = 0 + 0 = 0, \\ \text{and} \quad & \\ &u(L,\ t) = v(L,\ t) + w(L,\ t) = 0 + 0 = 0. \end{aligned}\right\} \quad (28.34)$$

so $u(x,\ t)$also satisfies the boundary conditions (28.2). Finally, we have

$$u(x,\ 0) = v(x,\ 0) + w(x,\ 0) = f(x) + 0 = f(x), \quad (28.35)$$

and

$$u_t(x,\ 0) = v_t(x,\ 0) + w_t(x,\ 0) = 0 + g(x) = g(x). \quad (28.36)$$

Thus $u(x,\ t)$ satisfies the general initial conditions (28.31) and (28.32).

**Exercises**

In Exercises 1-8, find the solutions $u(x,\ y)$ of the following equations by separating variables (product method).

1. $u_x + u_y = 0$
2. $u_x - yu_y = 0$
3. $u_x + u_y = 2(x + y)u$
4. $u_{xy} - u = 0$

5. $u_x - u_y = 0$
6. $u_{xx} + u_{yy} = 0$
7. $x^2 u_{xy} - 3y^2 u = 0$
8. $xu_x - yu_y = 0$
9. $xu_{xx} + 2u_t = 0$
10. $tu_{xx} + xu_t = 0$
11. $xu_{xx} + u_{xt} + u_t = 0$
12. $u_{xx} + u_{yy} + xu = 0$
13. $u_{xx} + (x+y)u_{yy} = 0$
14. $[p(x)u_x]_x - r(x)u_{tt} = 0$

In Exercises 15-18, find the deflection $u(x,\ t)$ of the vibrating string (length $L = \pi$ends fixed, and$a^2 = \frac{T}{\rho} = 1$) corresponding to zero initial velocity and initial deflection.

15. $0.02 sinx$
16. $k(\sin x - \sin 2x)$
17. $k(\pi x - x^2)$
18. $k(\pi^2 x - x^3)$

In Exercises 19-20, find the deflection $u(x,\ t)$ of the vibrating string (length$L = \pi$, ends fixed, and$a^2 = \frac{T}{\rho} = 1$) if the initial deflection $f(x)$ and the initial velocity $g(x)$ are:

19. $f(x) = 0, \quad g(x) = 0.1 \sin 2x$

20. $f(x) = 0.1 \sin x, \quad g(x) = -0.2 \sin x$

21. Solve the one dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

subject to the boundary conditions

$$u(0, t) = 0, u(l,\ t) = 0$$

and the initial conditions

$u(x,\, 0) = f(x)$ and $\frac{\partial u}{\partial t}\Big|_{t=0} = g(x)$ for all $t$,

where $f(x)$ is a given function and $l$ is a constant.

22. A string is stretched and fastened to two points $l$ apart. Motion is started by displacing the string in the form $u = a \sin \frac{\pi x}{l}$ from which it is released at time $t = 0$. Show that the displacement at any point at a distance $x$ from one end at time $t$ is given by

$$u = u(x,\ t) = a \sin \frac{\pi x}{l} \cos \frac{\pi c t}{l}.$$

**Hint to the Exercise* 22: Given the initial displacement $u(x, 0) = a \sin \frac{\pi x}{l}$, initial velocity (i.e. velocity at time $t = 0$ is $\frac{\partial u}{\partial t}\Big|_{t=0} = 0$.

***Answers***

1. $u(x, y = ke^{c(x-y)}$
2. $u(x, y) = ky^c e^{cx}$
3. $u(x, y) = ke^{x^2+y^2+c(x-y)}$
4. $u(x, y) = ke^{cx+\frac{y}{c}}$

9. $xF'' - kF = 0,\ 2\dot{G} + kG = 0$ with $u(x,\ t) = F(x)G(t)$

10. $F'' - kxF = 0,\ \dot{G} + k\,t\,G = 0$ with $u(x,\ t) = F(x)G(t)$

11. $F'' - k(F' + F) = 0,\ \dot{G} + kG = 0$ with $u(x,\ t) = F(x)G(t)$

12. $F'' + (x + k)\,F = 0,\ \ddot{G} - kG = 0$ with $u(x,\ y) = F(x)G(y)$

13. Not separable

14. $[p(x)F']' + k\,r(x)\,F = 0,\ \ddot{G} + kG = 0$ with $u(x,\ t) = F(x)G(t)$

15. $u(x,t) = 0.02\cos t \sin x$

16. $u(x,t) = k(\cos t \sin x - \cos 2t \sin 2x)$

17. $u(x,t) = \frac{8k}{\pi}\left[\cos t \sin x - \frac{1}{3^3}\cos 3t \sin 3x + \ .\ \ .\ \ .\ \right]$

18. $u(x,t) = 12k\left[\cos t \sin x - \frac{1}{2^3}\cos 2t \sin 2x + \ .\ \ .\ \ .\ \right]$

# Syllabus

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## DIFFERENTIAL EQUATIONS

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### Aims, Objectives and Outcomes

Differential equations model the physical world around us. Many of the laws or principles governing natural phenomenon are statements or relations involving rate at which one quantity changes with respect to another. The mathematical formulation of such relations (modelling) often results in an equation involving derivative (differential equations). The course is intended to find out ways and means for solving differential equations and the topic has wide range of applications in physics, chemistry, biology, medicine, economics and engineering.

On successful completion of the course, the students shall acquire the following skills/knowledge.

- Students could identify a number of areas where the modelling process results in a differential equation.
- They will learn what an ODE is, what it means by its solution, how to classify DEs, what it means by an IVP and so on.
- They will learn to solve DEs that are in linear, separable and in exact forms and also to analyse the solution.
- They will realise the basic differences between linear and non linear DEs and also basic results that guarantees a solution in each case.
- They will learn a method to approximate the solution successively of a first order IVP.
- They will become familiar with the theory and method of solving a second order linear homogeneous and nonhomogeneous equation with constant coefficients.
- They will learn to find out a *series* solution for homogeneous equations with variable coefficients near *ordinary points.*
- Students acquire the knowledge of solving a differential equation using Laplace method which is especially suitable to deal with problems arising in engineering field.
- Students learn the technique of solving *partial differential equations* using the method of separation of variables

***Syllabus***

Text: Elementary Differential Equations and Boundary Value Problems (11/e): William E Boyce, Richard C Diprima And Douglas B Meade John Wiley and Sons(2017) ISBN: 1119169879

**Module-I (22 hrs)** 1.1: Some Basic Mathematical Models; Direction Fields

1.2: Solutions of some Differential equations

1.3: Classification of Differential Equations

2.1: Linear Differential Equations; Method of Integrating Factors

2.2: Separable Differential Equations

2.3: Modelling with First Order Differential Equations

2.4: Differences Between Linear and Nonlinear Differential Equations

2.6: Exact Differential Equations and Integrating Factors

2.8: The Existence and Uniqueness Theorem (proof omitted)

**Module-II (23 hrs)**

3.1: Homogeneous Differential Equations with Constant Coefficients

3.2: Solutions of Linear Homogeneous Equations; the Wronskian

3.3: Complex Roots of the Characteristic Equation

3.4: Repeated Roots; Reduction of Order

3.5: Nonhomogeneous Equations; Method of Undetermined Coefficients

3.6: Variation of Parameters

5.2: Series solution near an ordinary point, part1

5.3: Series solution near an ordinary point,part2

**Module-III (15 hrs)**

6.1: Definition of the Laplace Transform

6.2: Solution of Initial Value Problems

6.3: Step Functions

6.5: Impulse Functions

6.6: The Convolution Integral

**Module-IV (20 hrs)**

10.1: Two-Point Boundary Value Problems

10.2: Fourier Series
10.3: The Fourier Convergence Theorem
10.4: Even and Odd Functions
10.5: Separation of Variables; Heat Conduction in a Rod
10.7: The Wave Equation: Vibrations of an Elastic String

**References:**

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3. C. Henry Edwards and David E. Penney: Elementary Differential Equations (6/e) Pearson Education, Inc. New Jersey (2008) ISBN 0-13-239730-7

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6. James C Robinson: An Introduction to Ordinary Differential Equations Cambridge University Press (2004)ISBN: 0-521-53391-0